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Geodesics and their mappings

Geodetické křivky a jejich zobrazení

Ph.D. THESIS

DISERTAČNÍ PRÁCE

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Abstract: The Ph.D. thesis focuses on the study of specific problems related to the theories of geodesics, rotary, geodesic, and almost geodesic mappings of manifolds with metric or affine structures. Surfaces and also (pseudo-) Riemannian and Kähler product spaces where geodesic bifurcations exist were constructed. The theory of rotary mappings and transformations was further developed. Besides, a counterexample to already-known results was constructed. Geodesic mappings of surfaces of revolution were studied in detail. Moreover, the fundamental equations of geodesic and almost geodesic mappings of certain special manifolds in the form of a closed Cauchy-type system of partial differential equations were derived.

Keywords: geodesic, isoperimetric extremal of rotation, rotary mapping, geodesic mapping, almost geodesic mapping, (pseudo-) Riemannian space, space with affine connection

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Abstrakt: Disertační práce je zaměřena na studium některých problémů spjatých s teoriemi geodetických křivek, rotačních, geodetických a téměř geodetických zobrazení variet s metrickými nebo afinními strukturami. Byly zkonstruovány plochy a také (pseudo-) Riemannovy a Kählerovy produktové prostory, na nichž existují geodetické bifurkace. Upřesnili jsme teorii rotačních zobrazení a transformací. Mimo jiné byl zkonstruován protipříklad k doposud známým výsledkům. Detailně byla studována také geodetická zobrazení rotačních kvadrik. Dále byly odvozeny fundamentální rovnice geodetických a téměř geodetických zobrazení některých speciálních variet ve tvaru uzavřeného systému parciálních diferenciálních rovnic Cauchyho typu.

Klíčová slova: geodetiká křivka, isoperimetrická extremála rotace, rotační zobrazení, geodetické zobrazení, téměř geodetické zobrazení, (pseudo-) Riemannův prostor, prostor s afinní konexí

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Declaration:

I hereby declare that I wrote the dissertation entitled *Geodesics and their mappings* on my own under the supervision of prof. RNDr. Josef Mikeš, DrSc., and I only used sources listed in the references.

In Olomouc: Signature:

I would like to express thanks to my supervisor Prof. RNDr. Josef Mikeš, DrSc. for his guidance and support during all four years of study and also his patience whilst writing the thesis.

Contents

Summary

This Ph.D. thesis is devoted to the study of specific problems related to the theories of geodesics, rotary, geodesic, and almost geodesic mappings of manifolds with metric and affine structures. The Introduction presents a brief overview of the results and works related to the subject of the thesis.

The First Chapter deals with bifurcations of geodesics, i.e., with questions about the existence of two or more geodesics passing through the same point in the same direction. The original result is proof of the existence of surfaces of revolution that admit these bifurcations. In response to this result, surfaces where bifurcation of closed geodesic exists were constructed. Furthermore, (pseudo-) Riemannian and Kähler product spaces that contain these bifurcations are also constructed.

The Second Chapter is dedicated to rotary mappings and transformations. The Riemannian space that admits rotary mapping and at the same time is not isometric with a surface of revolution is found. This is a counterexample to known results. Another original result is a description of a method that determines whether the rotary vector field exists in (pseudo-) Riemannian space. The existence of this vector field yields the existence of rotary mapping onto (pseudo-) Riemannian spaces. The chapter is completed with a detailed study of rotary transformation.

The Third Chapter includes both geodesic and almost geodesic mappings. The geodesic transformations of quadric surfaces of revolution are studied in detail. Finally, the main equations of geodesic and almost geodesic mappings between some special manifolds are found in the form of a Cauchy-type system of partial differential equations.

Results presented in the Ph.D. thesis were included, especially in monograph J. Mikeš, L. Rýparová et al. *Differential geometry of special mappings*, Palacký University, Olomouc, 2019.

Anotace

Disertační práce je zaměřena na studium některých problémů spjatých s teoriemi geodetických křivek, rotačních, geodetických a téměř geodetických zobrazení variet s metrickými nebo afinními strukturami. Úvodní část představuje stručný přehled výsledků a prací souvisejících s předmětem disertační práce.

První kapitola disertační práce se zabývá geodetickými bifurkacemi, tj. otázkami existence více geodetik procházejících daným bodem ve stejném směru. Původním výsledkem je důkaz existence rotačních ploch, které připouštějí geodetické bifurkace. V návaznosti na získané výsledky jsou zkonsturovány plochy, na kterých existují bifurkace uzavřených geodetik. Dále jsou zkonstruovány (pseudo-) Riemannovy a Kählerovy produktové prostory, v nichž existují geodetické bifurkace.

Druhá kapitola je věnována rotačním zobrazením a transformacím. Jsou zde nalezeny Riemannovy prostory, které připouštějí rotační zobrazení a zároveň nejsou izometrické rotačním plochám. Toto je protipříklad k doposud známým výsledkům. Dalším výsledkem je popis metody, pomocí které lze určit, zda v (pseudo-) Riemannově prostoru existují rotační vektorová pole. Z existence zmiňovaných vektorových polí pak vyplývá i existence rotačních zobrazení na některé (pseudo-) Riemannovy prostory. Kapitola je ukončena studiem rotačních transformací.

Třetí kapitola pojednává o geodetických a téměř geodetických zobrazeních. Studují se zde geodetická zobrazení rotačních kvadrik. Jsou odvozeny rovnice geodetických a téměř geodetických zobrazení mezi speciálními varietami ve tvaru uzavřeného systému parciálních diferenciálních rovnic Cauchyho typu.

Výsledky předložené v této práci byly zařazeny zejména do monografie J. Mikeš, L. Rýparová et al. *Differential geometry of special mappings*, Palacký University, Olomouc, 2019.

I N T R O D U C T I O N

Study subjects of contemporary differential geometry

Geometric structures, their mappings, and transformations on smooth manifolds represent the crucial part of contemporary differential geometry. Mainly, modern differential geometry focuses on mappings and transformations which preserve certain properties of geometric objects, known as properties that are invariant under these mappings and transformations.

As stated below, geodesics have great importance in differential geometry and its applications. The history of geodesics goes back to the 18th century to the work of J. Bernoulli and L. Euler. Next, geodesic mappings are known from the works of E. Beltrami [5,6] and T. Levi-Civita [56]. The study of geodesic mappings was followed by a study of almost geodesic and rotary mappings dated back to the 20th century.

Geodesic, almost geodesic, and rotary mappings are special diffeomorphisms that map any geodesic of the first space onto geodesic, almost geodesic, and isoperimetric extremal of rotation, respectively, see [47,56,71,95]. These topics are studied in detail in the monograph [R1].

This Ph.D. thesis is dedicated to the topics mentioned in the previous paragraph. The main subject of this thesis is a study of geodesic bifurcations, rotary, geodesic, and almost geodesic mappings of (pseudo-) Riemannian spaces and spaces with affine connections.

Geodesics were first mentioned by Johann I. Bernoulli in his letter to Leibniz in 1697. Details are described in monograph [R1, pp. 119–121]. There are many textbooks devoted to the theory of geodesics, but geodesics also appear in many papers and special monographs about modern Differential Geometry, i.e. [1, 15, 18, 21, 22, 24, 43, 44, 56, 74, 80, 82, 83, 89, 95, 103].

The First Chapter of the dissertation deals with a question about the existence of geodesics. More precisely, there is an example of geodesic bifurcation, which can be described as a situation where two or more geodesics pass through a given point in a given direction, [R1, pp. 44–46], [R2, R5, R12], see also [1, 103].

Rotary mappings were introduced by Leiko in [47]; these are mappings that take any geodesic of one two-dimensional Riemannian space onto the isoperimetric extremal of rotation of another two-dimensional Riemannian space. Leiko and others published many papers on rotary mappings and related problems [45–55]. In particular, they pointed out the importance of rotary mappings for physical applications [45, 49, 55]. Leiko [51] also studied infinitesimal rotary transformations, for other examples of infinitesimal transformations see [31, 70, 81, 82, 106–110]. Mikeš, Sochor, Stepanova [79] found new equations of isoperimetric extremals of rotation and Chudá, Mikeš, Sochor [16] also presented a more generalized definition of rotary mappings.

The Second Chapter of the dissertation is dedicated to a further study of rotary mappings. For example, we have constructed a counterexample to known results and found conditions of the existence of the rotary mappings, [R1, pp. 137–148], [R4, R6, R7, R9, R10].

The Third Chapter includes new results in the theory of geodesic and almost geodesic mappings. We found the equations of geodesic and almost geodesic mappings between special manifolds, [R3, R11, R13, R14]. Further, we briefly present the theory of geodesic and almost geodesic mappings. There are many different approaches to this theory, as well as many results.

Geodesics and their generalizations are essential, especially in geometric structures studies. Let us mention E. Beltrami [5, 6] and his contribution to non-Euclidean geometry. Mappings that preserve geodesics were also studied during Beltrami's time, later known as *geodesic mappings*. T. Levi-Civita [56] provided some interesting applications of geodesic mappings in terms of dynamic processes modelling in mechanics. With this work, he contributed to the basis of geodesic mappings in tensor form.

T. Levi-Civita [56], T. Thomas [105], H. Weyl [112] significantly contributed to the theory of general properties and dependencies of geodesic mappings. T. Levi-Civita found the equations describing geodesic mappings of Riemannian spaces and also found the metric form of Riemannian spaces that admit geodesic mappings. T. Thomas and H. Weyl found invariant objects of geodesic mappings. It is a fact that any space with affine connection is locally projectively equivalent to some equiaffine space. These "standard" results are stated and further developed in monographs by L. P. Eisenhart [21, 22].

N. S. Sinyukov [95] derived fundamental equations of geodesic mappings of Riemannian spaces, in a linear form, and thereby found conditions for the existence of the geodesic mapping. He also proved that the main equations for geodesic mappings of (pseudo-) Riemannian spaces are equivalent to some linear Cauchy-type system of differential equations in covariant derivatives. J. Mikeš and V. E. Berezovskii [72] dealt with the same problem for geodesic mapping of spaces with an equiaffine connection onto Riemannian spaces. This result holds even for spaces with a general affine connection. It follows from the study of É. Cartan [13] and J. M. Thomas [104], see [22, p. 105]. In 1924, they found a simple construction of projectively equivalent equiaffine connection, so-called "normal" connection. Therefore, the result of Eastwood and Matveev [19] is trivial.

Following the study of geodesic mappings, the term *degree of mobility* of Riemannian manifolds with respect to geodesic mappings was defined. It is a number of real parameters on which a solution of a Cauchy-type system of partial differential equations depends. Here, the existence of the solution of the system yields the existence of geodesic mapping. In [42, 60, 68], the authors found a connection between the degree of mobility with respect to geodesic mappings and degrees of isometric, homothetic, and projective transformations.

The geodesic mappings theory focuses on geodesic mappings of special spaces, namely spaces with constant curvature, Einstein, equidistant, symmetric, semi-symmetric, recurrent spaces, and their generalizations, see [2, 3, 8, 10, 11, 21, 26–30, 44, 56–59, 61–65, 67, 68, 70, 73, 76–78, 80, 84, 95, 96, 113]. Geodesic deformations were studied by M. L. Gavrilčenko, J. Mikeš, and others [23, 68, 70, 80, 90].

Finally, let us mention a study of geodesic mappings of spaces with affine connection, Finsler, and even more general spaces, i.e. [68, 70, 80, 86, 88, 91, 95, 105, 112].

Many monographs are dedicated to generalizations of geodesic mappings, such as almost geodesic mappings, holomorphically-projective mappings, F-planar mappings, conformally-projective mappings, and also their transformations and deformations.

The class of almost geodesic mappings is a natural generalization of the class of geodesic mappings. N. S. Sinyukov [95] introduced mapping, which maps any geodesic onto almost geodesic and entitled it *almost geodesic mapping*. He also established three types of almost geodesic mappings and denoted them π_1, π_2, and π_3. Later, V. Berezovskii and J. Mikeš [9,69] specified this classification and proved that no other type than those three mentioned above exists.

In 1962, A. Z. Petrov [85] studied quasi-geodesic mappings and showed that they could be used to simulate physical processes and electromagnetic fields. Comparable results are presented in the paper of C. L. Bejan and O. Kowalski [4]. The mappings $\pi_2(e)$ are similar to those mentioned above. All these spaces are connected with some affinor structure F, which can be interpreted as a force field.

The theory of almost geodesic mappings was developed by V. S. Shadnyi [92] and many others [86–88, 99, 100, 111]. Almost geodesic mappings of symmetric spaces were studied by V. S. Sobchuk [97] and V. S. Sobchuk, J. Mikeš, and O. Pokorná [98].

Nowadays, the theories of geodesics, geodesic mappings, and their generalizations were developed in many directions, i.e. in works [19, 20, 32–36, 41, 101, 102].

The aim of the Ph.D. thesis

This Ph.D. thesis aims to study the existence of geodesic bifurcations. It also further develops results in the theory of rotary mappings and transformations. Other subjects of study are geodesic and almost geodesic mappings.

The main aims are:

- a detailed study of geodesic bifurcations and certain problems related to this topic;
- a detailed study of rotary mappings and transformations of (pseudo-) Riemannian spaces and the analysis of their equations;
- a study of geodesic and almost geodesic mappings of some special spaces.

Methods

We use classical tensor calculus for Riemannian and pseudo-Riemannian manifolds and manifolds with affine connection, in local form, as well as in global form.

Results

I. Geodesic bifurcations; [R1, pp. 44–46, 186–187], [R2, R5, R12]

- A surface of revolution that admits geodesic bifurcations was found.
- Based on the previous result, the bifurcation of closed geodesics was constructed.
- We constructed n-dimensional Riemannian and Kähler product spaces that admit local geodesic bifurcations and also bifurcations of closed geodesic.

II. Rotary mappings and transformations; [R1, pp. 437–448], [R4, R6, R7, R9, R10]

- We proved that there exist spaces which are not isometric to surfaces of revolution and concurrently admit rotary mappings.
- We performed an in-depth analysis of rotary vector field equations and obtained fundamental equations in the Cauchy-type form.
- We proved that any surface of revolution with differentiable Gaussian curvature admits rotary mappings.
- Lastly, we studied infinitesimal rotary transformations.

III. Geodesic and almost geodesic mappings; [R3, R11, R13, R14]

- We proved that under geodesic transformation, quadric surfaces of revolution map onto surfaces of revolution, which are no longer quadric surfaces. Furthermore, these surfaces are bounded in space.
- Additionally, we dealt with geodesic mappings of spaces with an affine connection onto generalized Ricci symmetric spaces. We obtained equations of these mappings in the form of a Cauchy-type system of equations in covariant derivatives. We specified the number of parameters for this system.
- We considered geodesic mappings of Riemannian spaces onto Ricci-2-symmetric Riemannian spaces. The main equations of the mappings were obtained as a closed Cauchy-type system of differential equations in covariant derivatives. We estimated the number of essential parameters on which the solution depends.
- We found conditions that must be fulfilled for the canonical almost geodesic mapping of $\pi_2(e)$ type to preserve Riemannian tensor. We derived the main equations of canonical almost geodesic mapping of $\pi_2(e)$ type in the form of a Cauchy-type system of partial differential equations (PDE's). The number of real parameters was estimated.

Theoretical benefits and applicability of the results

The results obtained in the dissertation extend already known facts in theory of geodesics, rotary mappings of (pseudo-) Riemannian spaces, and also in the theory of geodesic and almost geodesic mappings of special spaces. Therefore, they are of theoretical importance in the differential geometry of special manifolds.

The results can be used in various applications, such as in dynamic processes in electromagnetic fields, in the modelling of physical fields, mechanics, thermodynamics, etc.

The Ph.D. thesis has a character of basic research.

Author's publications related to the Ph.D. thesis

[R1] Mikeš J., Rýparová L. et al.: *Differential geometry of special mappings*, Olomouc: Palacký University, 2019, 676 p.
ISBN 978-80-244-5535-8, DOI 10.5507/prf.19.24455358

[R2] Rýparová L., Mikeš J.: *On geodesic bifurcations*, Geometry, Integrability and Quantization 18, 2017, 217–224.
DOI 10.7546/giq-18-2017-217-224 (Scopus, WoS)

[R3] Rýparová L., Mikeš J.: *On global geodesic mappings of quadrics of revolution*, Proc. 16th Int. Conf. Aplimat, 2017, 1342–1348. (Scopus)

[R4] Rýparová L., Mikeš J.: *On rotary mappings of surfaces of revolution*, Proc. of 4th Conf. MITAV, 2017, Brno, 208–216.

[R5] Rýparová L., Mikeš J.: *Bifurcation of closed geodesi*c, Geometry, Integrability and Quantizations 19, 2018, 188–192.
DOI 10.7546/giq-19-2018-188-192 (Scopus, WoS)

[R6] Mikeš J., Rýparová L., Chudá H.: *On the theory of rotary mappings*, Math. Notes, 104(3–4), 2018, 617–620.
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[R7] Rýparová L., Mikeš J., Křížek J.: *On fundamental equations of rotary vector field*, Proc. 18th Int. Conf. Aplimat, 2019, 1031–1035.
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[R8] Mikeš J., Berezovski V., Peška P., Rýparová L.: *On canonical F-planar mappings of spaces with affine connectio*n, Filomat 33:4, 2019, 1273–1278.
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DOI 10.2298/FIL1914475B (IF=0.789, Q2)

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[R13] Berezovski V.E., Cherevko Y., Rýparová L.: *Conformal and geodesic mappings onto some special spaces*, Mathematics 7:8, 664, 2019.
DOI 10.3390/math7080664 (IF=1.105, Q1)

[R14] Berezovski V., Mikeš J., Rýparová L., Sabykanov A.: *On canonical almost geodesic mappings of type* $\pi_2(e)$, Mathematics 8:1, 54, 2020.
DOI 10.3390/math8010054 (IF=1.105, Q1)

I.

C H A P T E R

GEODESIC BIFURCATIONS

This chapter is dedicated to questions about geodesic bifurcations. In other words, whether there exists more than one geodesic passing through a given point in a given direction.

We find examples of geodesic bifurcations on the surfaces but also on n-dimensional Riemannian and Kähler spaces.

1.1 Geodesics

The term *geodesic* is one of the most important terms in differential geometry theory but also in differential geometry applications, i.e. in cartography, mechanics, physics, and gravitational theory.

Geodesics are a natural generalization of Euclidean lines in Riemannian spaces and also spaces with affine connections. The theory of geodesic can be demonstrated in the theory of relativity: geodesic are trajectories of the freely moving object in the curvilinear space, and they replace the linear motion in Euclidean space. Currently, we can encounter geodesic theory in global positioning system GPS, and the physical and mathematical value of this theory is also well known since J. Bernoulli, L. Euler, and J. Lagrange.

In the present time, there exist different definitions of geodesics. We are going to use a "classical" definition, which is stated in [21, 22, 56, 70, 74, 82, 84, 95].

Let $A_n = (M, \nabla)$ be an n-dimensional manifold M with an affine (or linear) connection ∇ and let $\gamma\colon I \to M$ be a regular curve on M.

Definition 1. A curve γ is called *geodesic* if there exists a tangent vector field parallel along it.

That means, for any value of a parameter $t_0 \in I$, where I is an interval, the tangent vector $\lambda(t_0)$ of the curve γ remains a tangent vector after any parallel transport. Therefore, the tangent vector $\lambda(t)$ of the geodesic γ is recurrent, thus it holds:

$$\nabla_{\lambda(t)}\lambda(t) = \rho(t)\lambda(t), \tag{1}$$

where $\rho(t)$ is a function defined on the interval I.

The geodesic is parametrized with *canonical (affine)* parameter s if the equation (1) can be rewritten in the form:

$$\nabla_{\lambda(s)}\lambda(s) = 0 \qquad \text{for all} \quad s \in I'. \tag{2}$$

That means the vector field $\lambda(s)$ is parallel along the geodesic.

Let us note that locally the canonical parameter of the geodesic always exists and arc length of Riemannian spaces is always canonical.

Rather often, we call geodesic those curves that satisfy equations (2).

In local chart (U, x) equation (2) has following form:

$$\ddot{x}^h(s) + \Gamma^h_{ij}\big(x(s)\big)\,\dot{x}^i(s)\,\dot{x}^j(s) = 0 \tag{3}$$

where $x^h = x^h(s)$ are equations of geodesic γ on chart (U, x), Γ^h_{ij} are components of affine connection ∇, also called Christoffel symbols of the second kind. Here and after we denote "$\cdot$" a derivative with respect to canonical parameter s.

The system (3) can be rewritten in the following form of ordinary differential equations of the first order with respect to unknown functions $x^h(s)$ and $\lambda^h(s)$:

$$\begin{aligned} \dot{x}^h(s) &= \lambda^h(s) \\ \dot{\lambda}^h(s) &= -\Gamma^h_{ij}\big(x(s)\big)\,\dot{x}^i(s)\,\dot{x}^j(s). \end{aligned} \tag{4}$$

Here $\lambda^h(s)$ denotes a tangent vector of the curve $\gamma(s)$ at the point $x^h(s)$.

Initial conditions of equations (4)

$$x^h(s_0) = x^h_0, \quad \text{and} \quad \lambda^h(s_0) = \lambda^h_0 \tag{5}$$

satisfy that geodesic goes through point x_0^h in direction λ_0^h ($\neq 0$).

From the general theory of ordinary differential equations follows that equations (4) with initial conditions (5) have a solution if $\Gamma_{ij}^h(x)$ are continuous functions. If functions $\Gamma_{ij}^h(x)$ are differentiable, then the solution of Cauchy problem (4) and (5) is *unique*. Moreover, the last-mentioned applies even if functions $\Gamma_{ij}^h(x)$ satisfy Lipschitz condition. Geometrically speaking, the initial data (3) mean that for any point it is possible to find a geodesic which passes through this point or even has the prescribed direction.

In [70, p. 89] was given an example of the space with an affine connection where the components are not differentiable but continuous functions. In this case, there exist one and only one geodesic passing through the given point x_0^h in the given direction λ_0^h.

Example 1. Let us consider a space A_n with affine connection ∇ defined in a certain coordinate neighbourhood (U, x) by its continuous components

$$\Gamma_{hh}^h = f^h(x^h(s)), \qquad h = 1, \dots, n,$$

the other components are vanishing. The solution of the Cauchy problem (3) and (4) was obtained in the following explicit form

$$\int_{x_0^h}^{x^h(s)} \exp\left(\int_{x_0^h}^{w} f^h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}w = \lambda_0^h\, s.$$

The solution mentioned above is unique and exists in the whole neighbourhood U and from this follows the geodesic passing through the point x_0^h in the direction λ_0^h is also unique. Let us remind that in this example, the components of the connection were continuous but not differentiable functions.

1.2 Example of geodesic bifurcations

We show bifurcation of geodesics on surfaces of revolution where two different geodesics go through the same point in the *same* direction. A different approach can be found in [103].

Here, the term bifurcation of geodesic stands for a situation where different geodesics connect two different points on the surface. That means these geodesics pass through the same point but do not have the same tangent vector.

Let $\mathcal{S}_2$ be a surface of revolution given by the equations:

$$x = r(u)\cos v, \quad y = r(u)\sin v, \quad z = z(u), \tag{6}$$

where v is a parameter from $(-\pi, \pi)$ and $u \in I \subset \mathbb{R}$, where $I = \langle u_1, u_2\rangle$.

In these equations we exclude meridian corresponding to $v = \pi$. Naturally, we also exclude "poles" which correspond to $r(u) = 0$.

Surface of revolution $\mathcal{S}$ given by equations (6) has following metric

$$\mathrm{d}s^2 = \left(r'^2(u) + z'^2(u)\right)\mathrm{d}u^2 + r'^2(u)\,\mathrm{d}v^2.$$

Assume u is a canonical parameter of the forming curve $(r(u), 0, z(u))$ then it holds $r'^2(u) + z'^2(u) = 1$. For this case, a metric of the surface $\mathcal{S}$ is

$$\mathrm{d}s^2 = \mathrm{d}u^2 + r^2(u)\,\mathrm{d}v^2.$$

Let us simplify the above and denote $f(u) \equiv r^2(u)$. Nonzero components of metric and inverse tensors are

$$g_{11} = g^{11} = 1, \quad g_{22} = \left(g^{22}\right)^{-1} = f(u).$$

Non-vanishing Christoffel symbols of the first kind are $\Gamma_{122} = \Gamma_{212} = \frac{1}{2}\,f'(u)$, $\Gamma_{221} = -\frac{1}{2}\,f'(u)$ and nonzero Christoffel symbols of the second kind are:

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}\,\frac{f'(u)}{f(u)}, \quad \Gamma^1_{22} = -\frac{1}{2}\,f'(u).$$

Further, let $u \equiv x^1$ and $v \equiv x^2$. We can rewrite the equations (3) of geodesics on surface $\mathcal{S}$ in the following form:

$$\ddot{u} = \frac{1}{2}\,f'(u)\,\dot{v}^2, \qquad \ddot{v} = -\frac{f'(u)}{f(u)}\,\dot{u}\,\dot{v}. \tag{7}$$

Since the parameter s is canonical (arc length) it implies that the tangent vector of these geodesics is unitary, i.e. the first integral applies:

$$\dot{u}^2 + f(u)\,\dot{v}^2 = 1. \tag{8}$$

Trivially, we verify that u-coordinate curves ($u = s$, $v =$ const, i.e. "meridian") are geodesic. In general, the same does not apply for the v-coordinates, v-curves are geodesic if and only if $f'(u) = 0$ (here and after, we will refer to them as *gorge circles*).

Further, let us study geodesics, which are none of the mentioned above. Suppose that $v(s) \neq 0$, i.e. $\dot{v}(s) \neq 0$. Then, we can rewrite the second equation in (7) in the form

$$\frac{\ddot{v}}{\dot{v}} = -\frac{f'(u)}{f(u)}\,\dot{u}.$$

After modification and integration by s we get

$$\dot{v} = \frac{C_1}{f(u)}, \quad C_1 \in \mathbb{R}. \tag{9}$$

Using (9), from (8) we get $\dot{u}^2 + f(u)\,\dfrac{{C_1}^2}{f^2(u)} = 1$, therefore

$$\dot{u} = \sqrt{1 - \frac{{C_1}^2}{f(u)}}. \tag{10}$$

Finally, the equations (9) and (10) determine a system of differential equations of the first order.

Now we construct an example of rotational surface $\mathcal{S}$, where bifurcation mentioned above exists.

Let us choose function

$$r(u) = \frac{1}{\sqrt{1 - u^{2\alpha}}} \quad \left(\Rightarrow f(u) = \frac{1}{1 - u^{2\alpha}}\right), \quad u \in (-1, 1).$$

On the one hand, function r has to be continuous to guarantee the existence of Christoffel symbols and the possibility to write equations of geodesics. On the other hand, Christoffel symbols can neither satisfy the Lipschitz condition nor can be differentiable functions.

Otherwise, there would be a unique solution of the system, thus a unique geodesic and bifurcation would not exist.

Theorem 1 ([R2]). *Geodesic bifurcations exist on the above-mentioned surface of revolution $\mathcal{S}$ given by the equation* (6) *for* $\alpha \in (0.5, 1)$.

Proof. The statement can be proved by the existence of geodesics given by the equations:

$$\begin{aligned} &I. && u = 0, \quad v = s; \\ &II. && u = \big((1-\alpha)\, s\big)^{\frac{1}{1-\alpha}}, \quad v = s - \frac{\big((1-\alpha)\, s\big)^{\frac{1+\alpha}{1-\alpha}}}{1+\alpha}. \end{aligned} \tag{11}$$

It is straightforward to verify that curves given by the equations (11) are geodesics. We prove it by the direct substitution in fundamental equations (9) and (10).

These two geodesics pass through the same point $(0, 0)$ and have the same tangent vector $(0, 1)$.

The first one shall be called a *trivial* geodesic (as mentioned above also called gorge circle) and the second one *non-trivial* geodesic passing through the point $(0, 0)$ in the direction $(0, 1)$.

We note a remarkable fact that the consequence of this statement is that geodesic bifurcation exists in each point of the gorge circle since the given surface of revolution is symmetric. Furthermore, there exist an infinite number of geodesics going through the point $(0, 0)$ in direction the $(0, 1)$.

Moreover, the surface $\mathcal{S}$ has the metric $\mathrm{d}s^2 = \mathrm{d}u^2 + r^2(u)\,\mathrm{d}v^2$, and since any surface with this metric admits a non-trivial geodesic mapping, then projective corresponding spaces preserve geodesic bifurcations mentioned above. □

Remark 1. If we set $f(u) = -\dfrac{1}{1-u^{2\alpha}}$, then metric of the surface $\mathcal{S}$ is indefinite and the equations (11) describe geodesic bifurcation on pseudo-Riemannian space.

1.3 Bifurcation of closed geodesics

The construction of a surface, where the bifurcation of closed geodesic would exist, is based on the results in the previous part. We consider a certain neighbourhood along the gorge circle of the surface $\mathcal{S}$ where bifurcations exist. First, let γ be a geodesic which starts at the point $(0, 0)$ and goes in the direction $(0, 1)$. Next, we suppose that this curve has its "end" also on the gorge circle and is also part of some non-trivial geodesic. That is possible because we create a surface of revolution thus it has to be symmetrical. The goal here is to connect those ends with the curve that would form the surface, i.e. lies on the surface of revolution.

Since we have some starting part of the curve γ, we can calculate functions $\dot{u}$ and $\dot{v}$ at the known point β, for $\beta > 0$. We denote canonical parameter corresponding to this point as s_β, then

$$\dot{u}(s_\beta) = \dot{u}_\beta \quad \text{and} \quad \dot{v}(s_\beta) = \dot{v}_\beta.$$

Therefore, after substitution in the equations (9) and (10) we obtain

$$\dot{u}_\beta = \sqrt{1 - \frac{C_1^2}{f(u_\beta)}}, \quad \dot{v}_\beta = \frac{C_1}{f(u_\beta)}.$$

Let us consider a certain point B, the *extremal* point, on the geodesic curve γ at which the tangent vector is the same as the tangent vector at the starting point but does not have to be unitary. Furthermore, we suppose that the point B also lies on the gorge circle where bifurcations exist. We take another neighbourhood along the gorge circle of the surface $\mathcal{S}$ and move along the axis z (axis of rotation), so the point B lies on the second gorge circle. Similarly, we denote functions corresponding to the point B as follows

$$\dot{u}(s_B) = \dot{u}_B \quad \text{and} \quad \dot{v}(s_B) = \dot{v}_B.$$

Again, after direct substitution in the equations (9) and (10) we get

$$\dot{u}_B = \sqrt{1 - \frac{C_1^2}{f(u_B)}}, \quad \dot{v}_B = \frac{C_1}{f(u_B)}. \tag{12}$$

As we mentioned above the tangent vector at the point B should have the same direction as vector $(0,1)$, thus

$$\dot{u}_B \equiv 0 = \sqrt{1 - \frac{C_1^2}{f(u_B)}}.$$

From it yields that

$$f(u_B) = C_1^2,$$

where C_1 is a real constant. We verify that the result also satisfies the second equation from (12)

$$\dot{v}_B = \frac{C_1}{f(u_B)} = \frac{C_1}{C_1^2} = C_1.$$

Trivially, we see that the vector $(0, C_1)$ has the same direction as the vector $(0,1)$.

The situation is illustrated in the Figure 1.

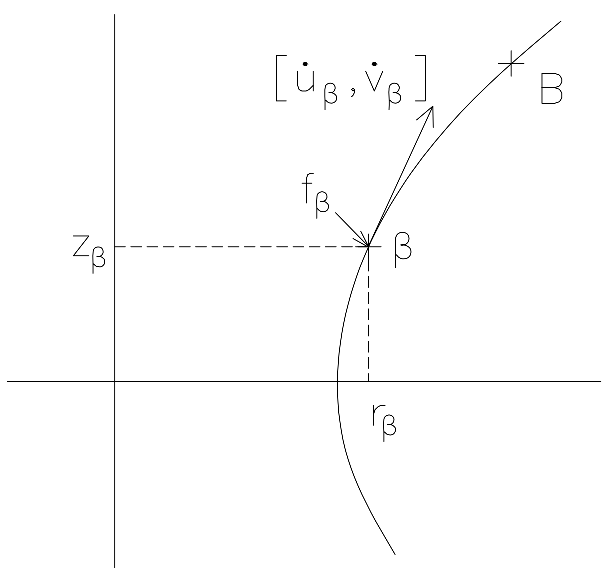

Figure 1: The graphical representation.

Finally, we have have two points A and B of the geodesic γ and certain neighbourhoods around these points where bifurcations exist. The rest of the geodesic is constructed as smooth curve that has to satisfy equations (9) and (10).

In the global scale, the situation is demonstrated in the Figure 2.

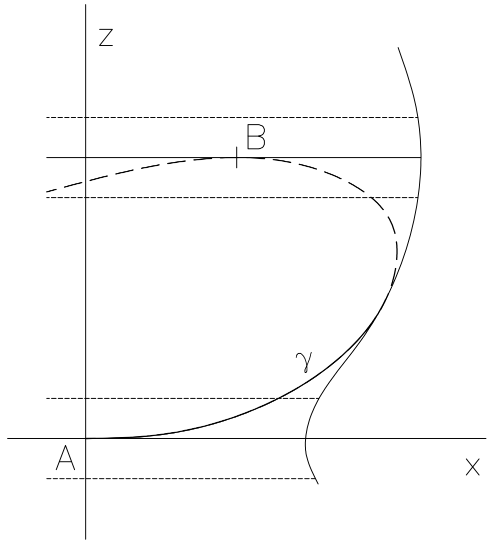

Figure 2: Global situation.

1.4 Geodesic bifurcation of product spaces

A Riemannian manifold $\mathbb{V}_n$ is called *product manifold* of the Riemannian manifolds $\overset{1}{\mathbb{V}}_{n_1}$, $\overset{2}{\mathbb{V}}_{n_2}$, ..., $\overset{m}{\mathbb{V}}_{n_m}$ $(n_1 + n_2 + \cdots + n_m = n)$

$$\mathbb{V}_n = \overset{1}{\mathbb{V}}_{n_1} \otimes \overset{2}{\mathbb{V}}_{n_2} \otimes \cdots \otimes \overset{m}{\mathbb{V}}_{n_m} \tag{13}$$

if the metrics are related by

$$g = g_1 \otimes g_2 \otimes \cdots \otimes g_m.$$

Locally, this means that there exists a coordinate system (x^i) such that the metric forms of these Riemannian manifolds satisfy

$$\mathrm{d}s^2 = \mathrm{d}s_1^2 + \mathrm{d}s_2^2 + \ldots + \mathrm{d}s_m^2,$$

where

$$\mathrm{d}s^2 = g_{ij}\,(x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j \quad \text{and} \quad \mathrm{d}s^2_\sigma = \overset{\sigma}{g}_{i_\sigma j_\sigma}(x^{k_\sigma})\,\mathrm{d}x^{i_\sigma}\,\mathrm{d}x^{j_\sigma}$$

$$i, j, k = \langle 1, n\rangle; \quad i_\sigma, j_\sigma, k_\sigma = \langle p_\sigma, r_\sigma\rangle$$

$$1 = p_1 \leq r_1 < p_2 \leq r_2 < \cdots < p_m \leq r_m = n.$$

It is known [70, pp. 192] that a product manifold $\mathbb{V}_n$ with metric

$$\bar{g} = (\alpha_1 g_1) \otimes (\alpha_2 g_2) \otimes \cdots \otimes (\alpha_m g_m), \qquad \text{where} \qquad \alpha_\sigma \neq 0$$

admits affine mappings $f : \mathbb{V}_n(M, g) \longrightarrow \overline{\mathbb{V}}_n(M, \bar{g})$.

By analysing equations (3), we can verify that the geodesic γ in product manifold $\mathbb{V}_n = (M, g)$ can be generated by geodesic $\overset{i}{\gamma} \subset \overset{i}{\mathbb{V}}_n$ for $i = 1, 2, \dots m$ like follows

$$\gamma = \overset{1}{\gamma} \otimes \overset{2}{\gamma} \otimes \cdots \otimes \overset{m}{\gamma},$$

while among these geodesics $\overset{i}{\gamma}$ there can exist "trivial geodesics" which are points in the space, i.e. they are defined by the equations $x^{k_i} = \underset{0}{x}^{k_i} = \text{const}$.

Kähler manifold K_n is a Riemannian space with a metric g and a structure F that satisfies following conditions

$$F^2 = -\mathrm{Id}, \qquad g(X, FX) = 0, \qquad \nabla F = 0,$$

for any vector X, where ∇ is Levi-Civita connection of the space.

From the one side, it is basic to verified that the product space of the Kähler spaces

$$K_n = \overset{1}{K} \otimes \overset{2}{K} \otimes \cdots \otimes \overset{m}{K}$$

is also Kähler space with structure which have analogical construction

$$F = \overset{1}{F} \otimes \overset{2}{F} \otimes \cdots \otimes \overset{m}{F}, \qquad \text{i.e.} \qquad F^{h_\sigma}_{i_\sigma} = F^{h}_{\sigma_i},$$

for $\sigma = 1, \dots, m$, and the other components of the structure F are vanishing.

From the other side, two dimensional Riemannian manifold is always Kähler, see [70, pp. 130], where the following holds

$$F_i^h = \varepsilon_{ij} g^{jh}, \qquad \varepsilon_{ij} = \sqrt{g_{11}g_{22} - g_{12}^2} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Next, we use the previous construction of the product spaces for a similar construction of geodesic bifurcation in n-dimensional Riemannian spaces and also for Kähler spaces.

Evidently, it is sufficient to take one of the components $\overset{i}{\mathbb{V}}$ of the product space (13), described above, as the space with metric (11). For example, if $\overset{1}{\mathbb{V}}$ is the space with the metric described by the equation (11) and this space has a geodesic bifurcation at the point $\overset{1}{\gamma}(0)$ then it is obvious that geodesic

$$\gamma = \overset{1}{\gamma} \otimes \overset{2}{\gamma} \otimes \cdots \otimes \overset{m}{\gamma}$$

has a geodesic bifurcation at the point $\gamma(0)$.

Using the same idea, we can construct n-dimensional Riemannian spaces where the geodesic bifurcations would exist. Similarly, it is possible to create n-dimensional Kähler spaces which also admit geodesic bifurcations.

Let us note that the geodesic bifurcations may not exist in other spaces $\overset{2}{\mathbb{V}}, \dots, \overset{m}{\mathbb{V}}$. In case there exists a geodesic bifurcation at the point $\overset{i}{\gamma}(0)$ of the space $\overset{i}{\mathbb{V}}$ then the image of the geodesic in the product space $\mathbb{V}_n$ would be much more complicated.

If we use the global results about geodesic bifurcations obtained and described in the paper [R5], where the bifurcations of closed geodesic are constructed, it is also possible to construct an n-dimensional Riemannian and Kähler manifolds where bifurcation of closed geodesic exist.

Moreover, based on the construction of the product spaces, we can also construct pseudo-Riemannian spaces with geodesic bifurcations of isotropic geodesics, i.e. geodesics which have vanishing length.

Example 2. Let $V = \overset{1}{\mathbb{V}} \otimes \overset{2}{\mathbb{V}}$ be product space of the spaces $\overset{1}{\mathbb{V}}$ and $\overset{2}{\mathbb{V}}$, and let

$$g_{i_1 j_1}, \quad i_1, j_1 = 1, 2; \qquad g_{i_2 j_2}, \quad i_2, j_2 = 1, 2$$

be the positive and the negative metric forms of spaces $\overset{1}{\mathbb{V}}$, $\overset{2}{\mathbb{V}}$ respectively.

In the Riemannian space $\overset{1}{\mathbb{V}}$ the geodesic $\overset{1}{\gamma}$ has a bifurcation at the point $\overset{1}{\gamma}(0)$ (space $\overset{1}{\mathbb{V}}$ can be the space from the previous subsections). Then the length of the vector is $|\overset{1}{\gamma}(0)| = 1$. Let us suppose that in Riemannian space $\overset{2}{\mathbb{V}}$ there exists a geodesic $\overset{2}{\gamma}$ for which $|\overset{2}{\gamma}(0)| = 1$. Then the geodesic γ for which

$$x^{i_1} = \underset{1}{x}^{i_1} \qquad \text{and} \qquad x^{i_2} = \underset{2}{x}^{i_2}$$

has tangent vectors λ^i for which

$$\lambda^{i_1} = \mathrm{d}\underset{1}{x}^{i_1}/\mathrm{d}t \qquad \text{and} \qquad \lambda^{i_2} = \mathrm{d}\underset{2}{x}^{i_2}/\mathrm{d}t.$$

Since $g_{i_1 j_1}$ is a positive form and $g_{i_2 j_2}$ is a negative form, it is evident that $|\lambda| = 0$ because

$$|\lambda| = \sqrt{g_{ij}\lambda^i\lambda^j} = \sqrt{g_{i_1 j_1}\lambda^{i_1}\lambda^{j_1} + g_{i_2 j_2}\lambda^{i_2}\lambda^{j_2}} = \sqrt{1-1} = 0.$$

This subsection further develops some results and ideas from the papers [R2], [R5]. The n-dimensional Riemannian product spaces and also Kähler product spaces which admit local geodesic bifurcations and also bifurcations of closed geodesics are constructed.

II.

C H A P T E R

ROTARY MAPPINGS AND TRANSFORMATIONS

This chapter is dedicated to the study of rotary mappings and transformations. These are direct generalizations of geodesic mappings and transformations. Here, it is a generalization of a geodesic variational problem and also a variational problem of geodesic mappings of surfaces.

2.1 Isoperimetric extremal of rotation

Leiko introduced the term of *isoperimetric extremal of rotation* on surfaces $\mathcal{S}_2$ and two-dimensional Riemannian spaces $\mathbb{V}_2$ with metric g, see [47]. These curves are defined as a solution of the special variational problem on Riemannian spaces.

Definition 2 ([47])**.** A curve ℓ: $x = x(t)$ on surface or on two-dimensional (pseudo-) Riemannian space is called an *isoperimetric extremal of rotation* if ℓ is an extremal of the functionals $\theta[\ell]$ and $s[\ell] = \text{const}$ with fixed ends.

Here

$$s[\ell] = \int_{t_0}^{t_1} |\lambda| \, \mathrm{d}t \quad \text{and} \quad \theta[\ell] = \int_{t_0}^{t_1} k(t) \, \mathrm{d}t,$$

where $k(t)$ is the curvature and $|\lambda|$ is the length of the tangent vector λ of ℓ.

In [47, 52] Leiko proved that a curve ℓ is an isoperimetric extremal of rotation if and only if its Frenet curvature k and Gaussian curvature K are proportional

$$k = c \cdot K,$$

where c is a constant. For special case $c = 0$ the equation represents a geodesic.
Mikeš, Stepanova and Sochor [79] derived new simpler form of equation of isoperimetric extremal of rotation

$$\nabla_s \lambda = c \cdot K \cdot F\lambda,$$

where c is a constant, s is the arc length, F is a tensor $\binom{1}{1}$ which satisfies the conditions

$$F^2 = -e \cdot Id, \quad g(X, FX) = 0, \quad \nabla F = 0.$$

For Riemannian manifold $\mathbb{V}_2$ is $e = +1$ and F is *complex structure* and for (pseudo-) Riemannian manifold is $e = -1$ and F is a *product structure*. This tensor F is uniquely defined (with respect to the sign) using skew-symmetric and covariantly constant discriminant tensor ε_{ij}, which is defined

$$F_j^h = g^{hi}\varepsilon_{ij}, \qquad \varepsilon_{ij} = \sqrt{|g_{11}g_{22} - g_{12}^2|} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

2.2 Rotary diffeomorphism

In work [46, 47] Leiko introduced the term *rotary mappings* between two-dimensional Riemannian spaces. Chudá, Mikeš and Sochor [16] later generalized this definition for the manifolds with affine connection.

Definition 3. A diffeomorfism f of two-dimensional manifold $\bar{\mathbb{A}}_2$ onto two-dimensional (pseudo-)Riemannian manifold $\mathbb{V}_2$ is called *rotary mapping* if any geodesic in $\bar{\mathbb{A}}_2$ is mapped onto isoperimetric extremal of rotation in $\mathbb{V}_2$.

Chudá, Mikeš and Sochor [16] formulated necessary and sufficient condition for the existence of the rotary mapping of the space with an affine connection $\bar{\mathbb{A}}_2$ onto the Riemannian space $\mathbb{V}_2$. This condition is the existence of a special torse-forming vector field θ (in $\mathbb{V}_2$) which satisfies

$$\nabla_X \theta = \theta \cdot (\Theta(X) + \nabla_X K/K) + \nu \cdot X \tag{14}$$

for any tangent vector X, where ∇ is the Levi-Civita connection on $\mathbb{V}_2$, K is the Gaussian curvature, ν is a function, the form Θ is defined as $\Theta(X) = g(\theta, X)$, and g is a metric of the space $\mathbb{V}_2$.

The condition mentioned above is also the necessary and sufficient condition of the existence of the rotary mapping between Riemannian spaces, see [47].

2.3 On spaces admitting rotary mappings

2.3.1 Counterexample of spaces admitting rotary mappings

Leiko [47] also proved that the existence of the vector field θ is a necessary condition for the existence of the rotary mapping between two-dimensional Riemannian spaces $\mathbb{V}_2$. The vector field θ has to satisfy the necessary condition (14). Locally, equations (14) can be rewritten as follows

$$\theta^h_{,i} = \theta^h(\theta_i + \partial_i \ln |K|) + \nu\delta^h_i. \tag{15}$$

In the paper [47], see also [48, 52, 54], Leiko claims the equations (14) imply that the space $\mathbb{V}_2$ is isometric with a surface of revolution. From the form of the formula (14), Leiko assumed that the vector field θ is concircular. Therefore, he obtained this result.

In the next part, we prove that the statement mentioned above is not valid, i.e. the following theorem holds (Mikeš, Rýparová, Chudá, a short form of this result is presented in [R6]).

Theorem 2. *There exists a Riemannian space* $\mathbb{V}_2$ *which is not isometric with a surface of revolution and where the vector field satisfying equations* (14) *exists.*

Proof. The existence of a vector field θ is the necessary condition for the existence of the rotary mapping between the manifold $\bar{\mathbb{A}}_2$ and (pseudo-) Riemannian space $\mathbb{V}_2$. This vector field is a special type of torse-forming vector field, which was defined by K. Yano [114] and which has to satisfy the condition (14).

Riemannian spaces $\mathbb{V}_2$ where such vector field exists are characterized in semi-geodesic coordinates x with a metric it the following form:

$$\mathrm{d}s^2 = \left(\mathrm{d}x^1\right)^2 + f(x^1, x^2) \cdot \left(\mathrm{d}x^2\right)^2. \tag{16}$$

It is known that this form of the metric always exists in any (pseudo-) Riemannian space $\mathbb{V}_2$. As it was mentioned above, this coordinate system is called a semi-geodesic coordinate system.

Now, we compute non-vanishing Christoffel symbols of the first and the second kind, and the Gaussian curvature K of $\mathbb{V}_2$:

$$\Gamma_{122} = \Gamma_{212} = 1/2\, f_1, \quad \Gamma_{221} = -1/2\, f_1, \quad \Gamma_{222} = 1/2\, f_2, \quad \text{and}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = 1/2\, \frac{f_1}{f}, \quad \Gamma^1_{22} = -1/2\, f_1, \quad \Gamma^2_{22} = 1/2\, \frac{f_2}{f},$$

$$K = \frac{R_{1212}}{g_{11}g_{22} - g^2_{12}} = -\frac{f_{11}}{2f} + \left(\frac{f_1}{2f}\right)^2,$$

here and further we denote $f_i = \partial_i f$, $\partial_i \equiv \partial/\partial x^i$ and analogically $f_{ij} = \partial_{ij} f$. To simplify this relation, we use substitution $F = f_1/f$, and thus we obtain

$$K = -1/2\, F_1 - 1/4\, F^2, \tag{17}$$

where similarly as above $F_1 = \partial_1 F$.

After lowering indices in (15) we get

$$\theta_{h,i} = \theta_h(\theta_i + \partial_i K/K) + \nu g_{hi}, \tag{18}$$

where $\theta_i = g_{i\alpha}\theta^i$.

From which follows $\theta^1 = \theta_1$, and additionally, we suppose that in chosen coordinate system $\theta^2 = \theta_2 = 0$ holds.

For indices $(hi) = (12)$ from (18) and after lowering indices we obtain $\partial_2\theta_1 = \theta_1 \cdot \partial_2 K/K$ and after integration we get

$$\theta_1 = \varkappa(x^1)K,$$

where $\varkappa$ is a function of variable x^1. Evidently, for $(hi) = (21)$ formula (18) is identity and for $(hi) = (11)$ and (22) we get following equations

$$\nu = \theta_{11} - \theta_1^2 - \theta_1\, \partial_i K/K \quad \text{and} \quad \nu = \frac{1}{2}\, \theta_1 \cdot f_1/f.$$

We merge these formulas and obtain following equation

$$\frac{\varkappa'}{\varkappa} - \varkappa \cdot K = \frac{1}{2} \cdot \frac{f_1}{f}.$$

Therefore, from (17) we get the equation

$$F' = -\frac{1}{2}\,F^2 + \frac{1}{\varkappa}\,F - 2 \cdot \frac{\varkappa'}{\varkappa^2}, \tag{19}$$

which is a differential equation called Riccati equation, see [40]. Here, symbol " $'$ " denotes a derivative with respect to variable x^1 and in these formulas x^2 is a parameter.

Equation (18) with a given function $\varkappa(x^1)$ is a well-known Riccati equation for an unknown function F; the variable x^2 is a parameter. A general solution of this equation depends on one arbitrary function of x^2, which plays the role of the constant of integration. Since $\partial_1 \ln |f| = F$, integration yields

$$f = c(x^2) \cdot \exp\left(\int F\, dx^1\right), \tag{20}$$

where $c(x^2)$ is a differentiable function.

The above considerations lead to the following theorem [R6].

Theorem 3. *Let $\mathbb{A}_2$ be a space with affine connection, $\mathbb{V}_2$ any (pseudo-) Riemannian space, and let f be a rotary mapping of $\mathbb{A}_2$ onto $\mathbb{V}_2$. Then, the local structure of the metric of $\mathbb{V}_2$ has the form* (16)*, where the function f satisfies conditions* (19) *and* (20)*.*

It is well known that if f can be represented as $f = a(x^1) \cdot c(x^2)$, then the space $\mathbb{V}_2$ is isometric to a surface of revolution. This condition holds, i.e., in the case $c(x^2) = \text{const}$, in which the integration constant in Eq. (16) is indeed a constant.

It is obvious that in the general case, the function $\exp\left(\int F\,\mathrm{d}\,x^1\right)$ depends on both variables x^1 and x^2. From which follows that the space $\mathbb{V}_2$ is generally not isometric to a surface of revolution.

More precisely, the space $\mathbb{V}_2$ is isometric to a surface of revolution if and only if there exists a concircular vector field ξ satisfying the equations $\nabla_X \xi = \varrho\, X$. Considering the integrability conditions for these equations, it is straightforward to verify that they have a nontrivial solution only in the case where the following partial differential equation in f holds:

$$\partial_2 K_2 - \partial_1 K_1 + \partial_1 K_2 \cdot (f K_1/K_2 - K_2/K_1) + {}^1\!/_2 \cdot F {K_2}^2/K_1 -$$

$$-{}^1\!/_2 \cdot f F K_1 + {}^1\!/_2 \cdot f_1 K - {}^1\!/_2 \cdot K_2 f_2/f = 0,$$

where, as above $K = -{}^1\!/_2\, F_1 - {}^1\!/_4\, F^2$ and $F = f_1/f$.

Obviously, functions satisfying the conditions in Theorem 3 do not generally satisfy this equation, which proves Theorem 2. □

2.3.2 On equations (19)

Using substitution $F = 2 \cdot u'/u$ we get a linear differential equation of the second order respective unknown function u

$$u'' = \frac{1}{\varkappa}\, u' - \frac{\varkappa'}{\varkappa^2}\, u. \tag{21}$$

The general solution of equation (21) can be written in the following form

$$u = C_1 u_1(x^1) + C_2 u_2(x^1),$$

where $C_1 = C_1(x^2)$ and $C_2 = C_2(x^2)$.

Let us assume that $\mathcal{U}(x^1)$ is a particular solution of differential equation (21). We substitute this solution into differential equation (21) and then we obtain

$$\varkappa' = -\frac{\mathcal{U}''}{\mathcal{U}}\, \varkappa^2 + \frac{\mathcal{U}'}{\mathcal{U}}\, \varkappa.$$

It is the Bernoulli differential equation, which can be transformed into an inhomogeneous linear differential equation using substitution $v = \dfrac{1}{\varkappa}$ like follows

$$v' = -\frac{\mathcal{U}'}{\mathcal{U}}\, v + \frac{\mathcal{U}''}{\mathcal{U}}.$$

The last equation can be solved using method of variation of parameters from which we obtain $v = \dfrac{\mathcal{U}'}{\mathcal{U}}$ therefore $\varkappa = \dfrac{\mathcal{U}}{\mathcal{U}'}$. From it follows that one from the solutions of the equation

(21) with a priori given $\varkappa(x^1)$ is

$$u = \mathrm{e}^{\int 1/\varkappa \, \mathrm{d}x^1}.$$

If the functions $\mathcal{U}$ and $\mathcal{V}$ are two solution of the differential equation (21) it is possible to form their Wronskian

$$W = \begin{vmatrix} \mathcal{U} & \mathcal{V} \\ \mathcal{U}' & \mathcal{V}' \end{vmatrix} = \mathcal{U}\mathcal{V}' - \mathcal{V}\mathcal{U}'.$$

Then, after differentiating W and using (21) for $\mathcal{U}$ and $\mathcal{V}$ we get

$$W' = \begin{vmatrix} \mathcal{U}' & \mathcal{V}' \\ \mathcal{U}' & \mathcal{V}' \end{vmatrix} + \begin{vmatrix} \mathcal{U} & \mathcal{V} \\ \mathcal{U}'' & \mathcal{V}'' \end{vmatrix} = \begin{vmatrix} \mathcal{U} & \mathcal{V} \\ \mathcal{U}'/\varkappa - \mathcal{U} \cdot \varkappa'/\varkappa^2 & \mathcal{V}'/\varkappa - \mathcal{V} \cdot \varkappa'/\varkappa^2 \end{vmatrix} = \frac{1}{\varkappa} W.$$

Because $W' = \dfrac{1}{\varkappa} W$ we get this relation

$$W = C_1 \cdot \mathrm{e}^{\int 1/\varkappa \mathrm{d}x^1}, \tag{22}$$

where C_1 is a constant of integration.

Because $\dfrac{1}{\varkappa} = \dfrac{\mathcal{U}'}{\mathcal{U}}$ then $\displaystyle\int \frac{1}{\varkappa} \, \mathrm{d}x^1 = \ln |\mathcal{U}|$ and from (22) we obtain

$$\mathcal{U}\mathcal{V}' - \mathcal{V}\mathcal{U}' = C_1 \cdot \mathrm{e}^{\ln |\mathcal{U}|},$$

therefore, we get a linear inhomogeneous differential equation $\mathcal{V}' = \dfrac{\mathcal{U}'}{\mathcal{U}} \mathcal{V} + C_1$.

Firstly, we solve the related homogeneous equation $\mathcal{V}' = \dfrac{\mathcal{U}'}{\mathcal{U}} \mathcal{V}$ and we get the solution $\mathcal{V} = C \cdot \mathcal{U}$, where C is a constant of integration.

Secondly, using the method of variation of parameters, we suppose that C is a function of the variable x^1 and then we obtain $C = \displaystyle\int \frac{C_1}{\mathcal{U}} \, \mathrm{d}x^1$ thus, the other partial solution of (21) is

$$\mathcal{V} = C_1 \cdot \mathcal{U} \cdot \int \frac{1}{\mathcal{U}} \, \mathrm{d}x^1 + C_2,$$

where C_2 is a constant of integration. As above $C_1 = C_1(x^2)$ and $C_2 = C_2(x^2)$.

Finally, if a certain particular solution of the equation (21) is known, it is possible to find the other particular solution, therefore, the general solution of this equation. From this follows that the vector field θ which satisfies the conditions (18) always exists. Generally speaking, the Riemannian space $\mathbb{V}_2$ given by the metric in the form (16) is not a surface of revolution and the Theorem 2 is valid.

2.3.3 On rotary vector field

In this part, we focus on so-called *rotary vector fields*. These are the vector fields that satisfy equation (14), which can be in local chart rewritten like follows (15). A vector field satisfying equations (15) does not exist in any space. We show a method that helps us find this vector field in $\mathbb{V}_2$ or prove its non-existence.

We rewrite equations (15) in the "covariant form", i.e. we find an expression of covariant derivative of $\theta_{i,j}$. Therefore, we obtain (18):

$$\theta_{i,j} = \theta_i(\theta_j + \partial_j K/K) + \nu g_{ij}. \tag{23}$$

These equations given in $\mathbb{V}_2$ form a system of PDE's for unknown functions $\theta_1(x)$, $\theta_2(x)$, and $\nu(x)$.

We find the integrability conditions of the equations (23). After covariant derivation of (23) in direction x^k we obtain the equations

$$\theta_{i,jk} = \theta_{i,k}(\theta_j + \partial_j K/K) + \theta_i(\theta_{j,k} + \partial_{jk} \ln|K|) + g_{ij}\nu_{,k}.$$

Now we substitute (23) to the previous equation, and we get

$$\begin{aligned}\theta_{i,jk} = \ & \theta_i(\theta_j + \partial_j K/K)(\theta_k + \partial_k K/K) + \nu g_{ik}(\theta_j + \partial_j K/K) + \\ & + \theta_i(\theta_j(\theta_k + \partial_k K/K) + \nu_{,k} g_{ij} + \partial_{jk}(\ln|K|)).\end{aligned}$$

We alternate the last formula with respect to indices j and k. Based on the Ricci identity it holds $\theta_{i,jk} - \theta_{i,kj} = \theta_\alpha R^\alpha_{ijk}$, where R^h_{ijk} are components of the Riemann tensor of $\mathbb{V}_2$. We obtain the integrability conditions

$$\theta_\alpha R^\alpha_{ijk} = g_{ik}\left(\left(\theta_j + \partial_j K/K\right)\nu - \nu_{,j}\right) - g_{ij}\left(\left(\theta_k + \partial_k K/K\right)\nu - \nu_{,k}\right) + \\ +\theta_i(\theta_j\partial_k K - \theta_k\partial_j K)/K.$$

It is known that for space $\mathbb{V}_2$ it holds $R^h_{ijk} = K(\delta^h_j g_{ik} - \delta^h_k g_{ij})$. Therefore, we get

$$g_{ik}\left(\nu\left(\theta_j + \partial_j K/K\right) - \nu_{,j} - K\theta_j\right) - g_{ij}\left(\nu\left(\theta_k + \partial_k K/K\right) - \nu_{,k} - K\theta_k\right) + \\ +\theta_i(\theta_j\partial_k K - \theta_k\partial_j K)/K = 0. \tag{24}$$

After contraction of (24) with g^{ij} ($||g^{ij}|| = ||g_{ij}||^{-1}$) we have

$$\nu_{,k} = \nu(\theta_k - \partial_k K/K) - K\theta_k - \theta_\alpha\theta_\beta g^{\alpha\beta}\partial_k K/K + \theta_k g^{\alpha\beta}\theta_\alpha\partial_\beta K/K. \tag{25}$$

It can be easily verified that the system of equations (23) and (25) forms a closed Cauchy-type system of PDE's in covariant derivatives respective unknown functions $\theta_i(x)$ and $\nu(x)$.

Finally, we obtain the following theorem, [R7].

Theorem 4. *A two-dimensional (pseudo-) Riemannian manifold $\mathbb{V}_2$ admits rotary vector field θ if and only if the following closed Cauchy-type system of PDE's in covariant derivatives has a solution with respect to functions $\theta_i(x)$ and $\nu(x)$:*

$$\begin{aligned} \theta_{i,j} &= \theta_i(\theta_j + \partial_j K/K) + \nu g_{ij}, \\ \nu_{,i} &= \nu(\theta_i - \partial_i K/K) - K\theta_i - \theta_\alpha\theta_\beta g^{\alpha\beta}\partial_i K/K + \theta_i g^{\alpha\beta}\theta_\alpha\partial_\beta K/K. \end{aligned} \tag{26}$$

For initial data (i.e. initial Cauchy conditions)

$$\theta_i(x_0) = \theta^0_i \quad \text{and} \quad \nu(x_0) = \nu_0, \tag{27}$$

where $x_0 \in \mathbb{V}_2 \in C^3$ the system (26) has at most one solution $\theta_i(x)$ and $\nu(x)$.

We can replace the condition $\mathbb{V}_2 \in C^3$ (i.e. the space is a 3-times differentiable space) with a weaker condition, that is $g \in C^3$ and the Gaussian curvature $K \in C^1$.

Further, let us note that from the conditions (27) follows that the general solution (26) depends on no more than 3 real parameters. These are, for example, numbers θ^0_i and ν_0.

2.4 On rotary mappings of surfaces of revolution

This subsection is devoted to the study of rotary mappings of surfaces of revolution and Riemannian manifolds which are isometric with these surfaces. These questions were studied by Leiko [45, 47, 48, 50, 52–54]. He proved that special surfaces of revolution admit rotary mapping [47].

Rýparová and Mikeš [R4] proved that any surface of revolution with differentiable Gaussian curvature admits rotary diffeomorphism and the same holds for (pseudo-) Riemannian spaces. These results have local validity.

Leiko [47] has studied rotary mappings of surfaces of revolution. In [47] the metric of the surface $\mathcal{S}_2$ has the form

$$\mathrm{d}s^2 = f(r)\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2. \qquad (28)$$

He analyzed the equations (15) and proved a theorem that vector fields θ exist in Riemannian space $\mathbb{V}_2$ if and only if $\mathbb{V}_2$ is isometric with the surface of revolution $\mathcal{S}_2$ and the metric of $\mathbb{V}_2$ has one from the following forms:

$$(\tilde{g}_{ij}) = \frac{f(r)}{A^2(B+\sqrt{f(r)})^2}\ \operatorname{diag}(f(r), r^2),$$
$$(\tilde{g}_{ij}) = B^2 f(r)\ \operatorname{diag}(f(r), r^2), \quad \text{where } A \neq 0 \text{ and } B \text{ are const},$$

above-mentioned was formulated in Theorem 2, see [47].

Note that the metric $\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + f(x^1)(\mathrm{d}x^2)^2$ of the surface of revolution $\mathcal{S}_2$ is more general than the metric in form (28).

In the following part, we shall prove that all surfaces of revolution admit rotary mapping. Moreover, there exists a rotary mapping between space $\bar{\mathbb{A}}_2$ and any Riemannian space $\mathbb{V}_2$, that is isometric with the surface of revolution $\mathcal{S}_2$, and also any (pseudo-) Riemannian space $\mathbb{V}_2$, whose metric has the form $\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + f(x^1)(\mathrm{d}x^2)^2$.

Note that all obtained results have local validity.

New results in theory of rotary mappings. Now, we prove that vector fields described in (15) exist in any Riemannian space $\mathbb{V}_2$ which is isometric with surface of revolution $\mathcal{S}_2$. Therefore, following theorem holds (Rýparová and Mikeš [R4]).

Theorem 5. *Let $\mathcal{S}_2$ be a surface of revolution and $\bar{\mathbb{A}}_2$ space with an affine connection. If Gaussian curvature K of the space $\mathcal{S}_2$ is differentiable then there exist a rotary mapping of $\bar{\mathbb{A}}_2$ onto $\mathcal{S}_2$.*

Proof. To prove this theorem, we use the results presented in [16]. Therefore, if we prove that vector fields satisfying condition (15) exist on any surface of revolution $\mathcal{S}_2$ then rotary mapping between $\bar{\mathbb{A}}_2$ and $\mathcal{S}_2$ exists. We choose the metric of the surface of revolution $\mathcal{S}_2$ in the following form:

$$\mathrm{d}s^2 = \left(\mathrm{d}x^1\right)^2 + f(x^1)\left(\mathrm{d}x^2\right)^2. \tag{29}$$

Therefore, the Christoffel symbols of the first kind are

$$\Gamma_{122} = \Gamma_{212} = \frac{1}{2} f'(x^1) \quad \text{and} \quad \Gamma_{221} = -\frac{1}{2} f'(x^1),$$

the others are vanishing. The Christoffel symbols of the second kind are

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}\,\frac{f'(x^1)}{f(x^1)} \quad \text{and} \quad \Gamma^1_{22} = -\frac{1}{2} f'(x^1).$$

We calculate the Gaussian curvature $\quad K = \dfrac{1}{4}\left(\dfrac{f'(x^1)}{f(x^1)}\right)^2 - \dfrac{1}{2}\,\dfrac{f''(x^1)}{f(x^1)}.$

Let us suppose that $\theta^h = a(x^1)\,\delta^h_1$, thus from (15) we obtain following equations

$$a'(x^1) = a(x^1)\cdot\left(a(x^1) + \frac{K'}{K}\right) + \nu(x^1) \quad \text{and} \quad \frac{1}{2}\,a(x^1)\,\frac{f'(x^1)}{f(x^1)} = \nu(x^1).$$

Next, we merge these equations and obtain the following relation

$$a' = a^2 + a\cdot\left(\frac{K'}{K} + \frac{1}{2}\frac{f'}{f}\right).$$

This equation is a Bernoulli differential equation. Using the substitution $u = 1/a$, we get the inhomogeneous linear ordinary differential equation

$$u' = -1 - u\cdot\left(\frac{K'}{K} + \frac{1}{2}\,\frac{f'}{f}\right).$$

Now, we use the method of variation of parameters to solve the equation and we get $u(x^1) = \dfrac{c(x^1)}{K\sqrt{f(x^1)}}$. From this follows that $c'(x^1) = -K\sqrt{f(x^1)}$, thus

$$u(x^1) = \frac{1}{K\sqrt{f(x^1)}}\left(-\int K\sqrt{f(x^1)}\partial x^1\right).$$

Considering Gaussian curvature K has a special form stated above, we obtain

$$u(x^1) = \frac{1}{K\sqrt{f(x^1)}}\left(C + \frac{f'(x^1)}{2\sqrt{f(x^1)}}\right), \quad \text{where } C = \text{const}.$$

Consequently, the function $a(x^1)$ has the following form

$$a(x^1) = \frac{2K \cdot f(x^1)}{f'(x^1) + 2C\sqrt{f(x^1)}}.$$

Since the function $a(x^1)$ has to be differentiable, so does the Gaussian curvature K of the surface S_2. Therefore, vector fields exist for any surface of revolution, and the theorem is proved. □

As was mentioned above, the metric (29) of the surface of revolution $\mathcal{S}_2$ is more general than the metric (28) used by Leiko in [47]. Unlike the metric (28), it includes gorge circles.

In the proof of Theorem 5 the metric is used in the form (29), therefore, Theorem 5 holds for any Riemannian space $\mathbb{V}_2$ which is isometric with surface of revolution $\mathcal{S}_2$. Moreover, it holds even for pseudo-Riemannian spaces which have indefinite metric, in case $f(x^1) < 0$.

The general solution of (15) in system (29) is therefore

$$\theta^h = a \cdot \delta^h_1 \quad \text{where} \quad a(x^1) = \frac{2K \cdot f}{f' + 2C\sqrt{f}}, \tag{30}$$

and depends on the real parameter C.

Let us consider a certain rotary mapping between Riemannian space $\mathbb{V}_2$ and the manifold with affine connection $\bar{\mathbb{A}}_2$ that corresponds to a certain real constant C. We denote such manifold as $\bar{\mathbb{A}}_2(C)$.

Theorem 6 ([R4]). *Manifolds with affine connections $\bar{\mathbb{A}}_2(C_\alpha)$ and $\bar{\mathbb{A}}_2(C_\beta)$, $C_\alpha \neq C_\beta$ are not geodesically equivalent.*

Proof. Suppose that $\bar{\mathbb{A}}_2(C_\alpha)$ and $\bar{\mathbb{A}}_2(C_\beta)$ are images of the Riemannian space $\mathbb{V}_2$ in the rotary mapping and let $C_\alpha \neq C_\beta$. Then for the affine connection components $\mathcal{T}^h_{ij}(x)$ and $\bar{\mathcal{T}}^h_{ij}(x)$ of these spaces, respectively, the following equations hold

$$\mathcal{T}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_{(i}\psi_{j)} + \theta^h \cdot g_{ij}$$

$$\bar{\mathcal{T}}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_{(i}\bar{\psi}_{j)} + \bar{\theta}^h \cdot g_{ij},$$

where ψ_i, θ^h and $\bar{\psi}_i$, $\bar{\theta}^h$ are solutions of rotary mappings $\mathbb{V}_2 \to \bar{\mathbb{A}}_2(C_\alpha)$ and $\mathbb{V}_2 \to \bar{\mathbb{A}}_2(C_\beta)$, respectively.

We subtract these equations and get

$$\bar{\mathcal{T}}^h_{ij}(x) - \mathcal{T}^h_{ij}(x) = \delta^h_{(i}\bar{\psi}_{j)} - \delta^h_{(i}\psi_{j)} + (\bar{\theta}^h - \theta^h) \cdot g_{ij},$$

therefore

$$\delta^h_{(i}\omega_{j)} + (\bar{\theta}^h - \theta^h) \cdot g_{ij} = 0.$$

For indices $h = 1$, resp. $h = 2$ we obtain

$$\delta^1_{(i}\omega_{j)} + (\bar{\theta}^1 - \theta^1) \cdot g_{ij} = 0, \quad \text{resp.} \quad \delta^2_{(i}\omega_{j)} = 0,$$

thus $\omega_2 = 0$, $\omega_1 = 0$. From $\omega_i = 0$ it follows that $\bar{\theta}^h = \theta^h$. Using (30), we obtain

$$\frac{2K \cdot f}{f' + 2C_\alpha\sqrt{f}} = \frac{2K \cdot f}{f' + 2C_\beta\sqrt{f}},$$

therefore, $C_\alpha = C_\beta$ which is a contradiction with assumption $C_\alpha \neq C_\beta$ thus the theorem is proved. □

2.5 Infinitesimal rotary transformation

This subsection is devoted to further study of a certain type of infinitesimal transformations of two-dimensional (pseudo-) Riemannian spaces, which are called rotary. An infinitesimal transformation is called *rotary* if it maps any geodesic in (pseudo-) Riemannian space onto an isoperimetric extremal of rotation in their principal parts in (pseudo-) Riemannian space. We study basic equations of the infinitesimal rotary transformations in detail and obtain the simpler fundamental equations of these transformations.

2.5.1 Basic definition of infinitesimal rotary transformation

First, let us define the term of the infinitesimal rotary transformation of two-dimensional (pseudo-) Riemannian spaces $\mathbb{V}_2$.

Let us consider an 2-dimensional (pseudo-) Riemannian space $\mathbb{V}_2$, with the Levi-Civita connection ∇. We denote $x = (x^1, x^2)$ a coordinate system on the space $\mathbb{V}_2$.

A curve ℓ: $x = x(t)$ is a geodesic if and only if $\nabla_t \lambda = \rho(t)\lambda$, which can be rewritten into a coordinate form

$$d\lambda^h/dt + \Gamma^h_{\alpha\beta}(x(t))\lambda^\alpha\lambda^\beta = \rho(t)\lambda^h, \tag{31}$$

where $\lambda(t) = dx(t)/dt$ is a tangent vector of ℓ, Γ^h_{ij} are components of the connection ∇, and $\rho(t)$ is a function of parameter t.

A curve $\bar{\ell}$: $x = \bar{x}(t)$ is an isoperimetric extremal of rotation if and only if the following equation holds $\nabla_s \bar{\lambda} = cK \cdot F\bar{\lambda}$, and in the coordinate form

$$d\bar{\lambda}^h/dt + \Gamma^h_{\alpha\beta}(\bar{x})\bar{\lambda}^\alpha\bar{\lambda}^\beta = cK(\bar{x}) \cdot F^h_\alpha(\bar{x})\bar{\lambda}^\alpha, \tag{32}$$

where c is a constant, K is the Gaussian curvature, $\bar{\lambda}(t) = d\bar{x}(t)/dt$ is a tangent vector of $\bar{\ell}$, and F is an affinor. In this case, parameter t is a length of the curve $\bar{\ell}$ and $\bar{\lambda}$ is a unit tangent vector.

An *infinitesimal transformation* of a (pseudo-) Riemannian space $\mathbb{V}_n$ is given with respect to the coordinates in this manner

$$\bar{x}^h = x^h + \varepsilon\,\xi^h(x), \tag{33}$$

where x^h are the coordinates of a certain point in V_n and $\bar{x}^h$ are the coordinates of its image under the infinitesimal transformation, ε is an infinitesimal parameter not depending on x^h, and ξ^h is a displacement vector.

If a certain object $\mathcal{A}$ of the space $\mathbb{V}_n$ depends on $x \in \mathbb{V}_n$ but also on the infinitesimal parameter ε, then the *principal part* of the object $\mathcal{A}$ is $\overset{0}{\mathcal{A}}(x)+\overset{1}{\mathcal{A}}(x)\varepsilon$ in the expansion of series with respect to the infinitesimal parameter ε

$$\mathcal{A}(x,\varepsilon) = \overset{0}{\mathcal{A}}(x) + \overset{1}{\mathcal{A}}(x)\varepsilon + \overset{2}{\mathcal{A}}(x)\varepsilon^2 + \cdots .$$

For our purposes, the curves obtained by the infinitesimal transformation of geodesics satisfy the equations of isoperimetric extremals of rotation (32) under the condition, that we dropped the terms containing higher powers of the infinitesimal parameter ε, i.e. $\varepsilon^2, \varepsilon^3, \dots$.

Definition 4. An infinitesimal transformation of the (pseudo-) Riemannian space $\mathbb{V}_2$ is called *rotary* if it maps any geodesic of the space $\mathbb{V}_2$ onto an isoperimetric extremal of rotation in their principal parts.

2.5.2 Basic equations of infinitesimal rotary transformation

We prove the following theorem [R10].

Theorem 7. *A differential operator $X = \xi^\alpha(x)\partial_\alpha$ $(\partial_\alpha = \partial/\partial x^\alpha)$ determines an infinitesimal rotary transformation of (pseudo-) Riemannian space $\mathbb{V}_2$ if and only if X satisfies*

$$\mathcal{L}_\xi \Gamma^h_{ij} = \delta^h_{(i}\psi_{j)} + \theta^h g_{ij}, \qquad \theta^h_{,i} = \theta^h(\theta_i + K_i/K) + \nu\delta^h_i, \tag{34}$$

where ψ_i is a covector, δ_i^h is the Kronecker delta, θ^h is a vector field, g is a metric tensor, K ($\neq 0$) is the Gaussian curvature, and $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ.

Proof. Let us consider an infinitesimal rotary transformation of (pseudo-) Riemannian space V_2 determined by the equations (33). Furthermore, let ℓ be a geodesic of the space V_2 given by the equations $x^h = x^h(t)$. In addition, let ℓ satisfy the equations (31). The curve $\bar{\ell}$ which corresponds to the curve ℓ under the infinitesimal rotary transformation (33) has the following equations

$$\bar{x}^h(t) = x^h(t) + \varepsilon\, \xi^h(x(t)). \tag{35}$$

The infinitesimal transformation (33) is rotary if $\bar{\ell}$ is an isoperimetric extremal of rotation in its principal parts. Therefore, the equations $\bar{x}(t)$ given by (35) satisfy in the principal part equations (32) which could be written like follows

$$\frac{d\bar{\lambda}^h(t)}{dt} + \Gamma^h_{\alpha\beta}(\bar{x}(t))\bar{\lambda}^\alpha(t)\bar{\lambda}^\beta(t) = cK(\bar{x}(t)) \cdot F^h_\alpha(\bar{x}(t))\bar{\lambda}^\alpha(t). \tag{36}$$

Next, we shall find the objects involved in the equations (36). The tangent vector $\bar{\lambda}^h(t)$ we receive after derivation of equations (35)

$$\bar{\lambda}^h(t) \equiv \frac{d\bar{x}^h(t)}{dt} = \frac{dx^h(t)}{dt} + \varepsilon \frac{\partial \xi^h(x(t))}{\partial x^\gamma}\frac{dx^\gamma(t)}{dt} = \lambda^h(t) + \varepsilon\lambda^\gamma(t)\partial_\gamma\xi^h(x(t)).$$

Also, for the connection Γ and the structure F we get

$$\Gamma^h_{\alpha\beta}(\bar{x}) = \Gamma^h_{\alpha\beta} + \varepsilon\frac{\partial\Gamma^h_{\alpha\beta}}{\partial x^\gamma}\xi^\gamma + \boxed{\varepsilon^2} \quad \text{and} \quad F^h_\alpha(\bar{x}) = F^h_\alpha + \varepsilon\frac{\partial F^h_\alpha}{\partial x^\gamma}\xi^\gamma + \boxed{\varepsilon^2}.$$

Furthermore, for the Gaussian curvature K it holds

$$K(\bar{x}) = K + \varepsilon\frac{\partial K}{\partial x^\gamma}\xi^\gamma + \boxed{\varepsilon^2}.$$

Finally, we expand the function ρ and the constant c like follows

$$\rho(t) = \rho_0(t) + \varepsilon\rho_1(t) + \boxed{\varepsilon^2}, \quad \text{and} \quad c = c_0 + \varepsilon c_1 + \boxed{\varepsilon^2}.$$

Let us remind, that here and after $\boxed{\varepsilon^2}$ stands for the terms containing higher powers of the infinitesimal parameter ε, which will be dropped later.

Now, we substitute the expressions mentioned above into the equation (36) and we get

$$\frac{d\lambda^h}{dt} + \varepsilon\left(\partial_{\alpha\beta}\xi^h\lambda^\alpha\lambda^\beta + \frac{d\lambda^\alpha}{dt}\partial_\alpha\xi^h\right) +$$
$$+(\Gamma^h_{\alpha\beta} + \varepsilon\partial_\gamma\Gamma^h_{\alpha\beta}\xi^\gamma + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)(\lambda^\beta + \varepsilon\lambda^\gamma\partial_\gamma\xi^\beta) =$$
$$= (c_0 + \varepsilon c_1 + \boxed{\varepsilon^2})(F^h_\alpha + \varepsilon\partial_\gamma F^h_\alpha\xi^\gamma + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)(K + \varepsilon\partial_\gamma K\xi^\gamma + \boxed{\varepsilon^2}).$$

Since we know that the curve ℓ is a geodesic, we use equations (31) to eliminate $\frac{d\lambda^h}{dt}$ from the expression above

$$-\Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + \lambda^h(\rho_0 + \varepsilon\rho_1 + \boxed{\varepsilon^2})+$$
$$+\varepsilon\left(\partial_{\alpha\beta}\xi^h\lambda^\alpha\lambda^\beta + \partial_\alpha\xi^h\Big(-\Gamma^\alpha_{\beta\gamma}\lambda^\beta\lambda^\gamma + \lambda^\alpha(\rho_0 + \varepsilon\rho_1 + \boxed{\varepsilon^2})\Big)\right) +$$
$$+(\Gamma^h_{\alpha\beta} + \varepsilon\partial_\gamma\Gamma^h_{\alpha\beta}\xi^\gamma + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)(\lambda^\beta + \varepsilon\lambda^\gamma\partial_\gamma\xi^\beta) =$$
$$= (c_0 + \varepsilon c_1 + \boxed{\varepsilon^2})(F^h_\alpha + \varepsilon\partial_\gamma F^h_\alpha\xi^\gamma + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)(K + \varepsilon\partial_\gamma K\xi^\gamma + \boxed{\varepsilon^2}).$$

The constant term (not depending on ε) and the linear term (with respect to ε) from the equation mentioned above vanishes, in which case we receive two following equations, the first one

$$\rho_0\lambda^h = c_0F^h_\alpha\lambda^\alpha K, \tag{37}$$

and the second one

$$\lambda^\alpha\lambda^\beta(\partial_{\alpha\beta}\xi^h - \Gamma^\gamma_{\beta\alpha}\partial_\gamma\xi^h + \Gamma^h_{\alpha\gamma}\partial_\beta\xi^\gamma + \Gamma^h_{\gamma\beta}\partial_\alpha\xi^\gamma + \partial_\gamma\Gamma^h_{\alpha\beta}\xi^\gamma) + \rho_1\lambda^h + \rho_0\lambda^\alpha =$$
$$= c_0(KF^h_\alpha\lambda^\gamma\partial_\gamma\xi^\alpha + K\lambda^\alpha\partial_\gamma F^h_\alpha\xi^\gamma + \lambda^\alpha F^h_\alpha\partial_\gamma K\xi^\gamma) + c_1KF^h_\alpha\lambda^\alpha.$$

From the equation (37) follows that $\rho_0 = c_0 = 0$, which can be substituted to the last equation. Furthermore, after using the definition of Lie derivative we obtain the following relation

$$\mathcal{L}_\xi\Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = -\rho_1\lambda^h + c_1KF^h_\alpha\lambda^\alpha. \tag{38}$$

These equations hold true for any point and any unit vector λ^h. From the equation mentioned above we obtain the equations

$$\mathcal{L}_\xi\Gamma^h_{ij} = \delta^h_{(i}\psi_{j)} + \theta^h g_{ij}, \tag{39}$$

where ψ_i is a covector and θ^h is a vector.

Now, let us substitute the equations (39) in (38) and we get

$$(\delta_i^h \psi_j + \delta_j^h \psi_i + \theta^h g_{ij})\lambda^i \lambda^j = -\rho_1 \lambda^h + c_1 K F_i^h \lambda^i. \tag{40}$$

After contracting formula (40) with $g_{h\alpha}\lambda^\alpha$ we obtain $\theta_\alpha \lambda^\alpha = -(\rho_1 + 2\psi_\alpha \lambda^\alpha)$, where $\theta_i =$ $= g_{i\alpha}\theta^\alpha$. Therefore, formula (40) has the form

$$\eta \theta^h = \theta_\alpha \lambda^\alpha \lambda^h + c_1 K \cdot F_\alpha^h \lambda^\alpha, \tag{41}$$

where $\eta = g_{ij}\lambda^i \lambda^j = \pm 1$. After differentiating (41) along the curve ℓ and after detailed analysis of degrees of λ^h in such equation, we get

$$\nabla \theta^h = \theta^h(\theta_i + \nabla K/K) + \nu \delta_i^h, \tag{42}$$

where ν is a function on the space $\mathbb{V}_2$, therefore the theorem is proved. □

As it can be seen, the equations (34) have a simpler form than the equations of rotary transformations deduced by Leiko in [47]. Here, Leiko stated that the vector field θ which satisfies equations (42) exist only on the surfaces of revolution. This statement is not valid and we have constructed a counterexample in [R6], see Theorem 2, p. 35.

III.

C H A P T E R

GEODESIC AND ALMOST GEODESIC MAPPINGS

Presented chapter deals with geodesic and almost geodesic mappings of special spaces. Term geodesic mappings is based on one of the most important terms of differential geometry, and that is *geodesic*, see p. 20.

The first ideas about this theory appeared in the 19th century in the paper [56] by T. Levi-Civita. Here, the author defined a problem of finding Riemannian spaces with common geodesics and solved it in a special coordinate system. Let us remark an interesting fact, that this theory was connected with a study of equations of the mechanical systems dynamic.

3.1 Geodesic mappings of surfaces of revolution

Many authors worked on geodesic mappings. For example, E. Beltrami studied geodesic mappings of spaces of constant curvature [6]. Also, U. Dini contributed to the theory of geodesic mappings of Euclidean spaces, see [17]. Levi–Civita [56] concerned with general properties of geodesic mappings of Riemannian spaces, as did many others [70, 80, 84, 95]. Also, many results about the general theory of geodesic mappings and deformations can be found in [70, 80, 84, 90, 95].

Many works are related to geodesic deformations, such as geodesic deformations of surfaces of revolution [62], globally geodesic deformation of a sphere [61] or global geodesic mappings and transformations of an ellipsoid [26]. Moreover, it was proved that many Riemannian spaces do not admit trivial geodesic mappings [59, 70, 80, 84, 90, 95].

In the next part, we present geodesic deformations of quadric surfaces of revolution. Mappings under consideration deform quadric surfaces to surfaces of revolution, which are no longer quadric surfaces.

3.1.1 Surfaces of revolution

A diffeomorphism between surfaces $\mathcal{S}_2$ and $\bar{\mathcal{S}}_2$ is called a *geodesic mapping* if any geodesic in $\mathcal{S}_2$ maps onto a geodesic in $\bar{\mathcal{S}}_2$. A *geodesic deformation* is a deformation of $\mathcal{S}_2$ which preserves geodesics.

Let $\mathcal{S}_2$ be a surface of revolution in the Euclidean space $\mathcal{E}_3$ which is in Cartesian coordinates (x, y, z) given by the equations

$$x = r(w)\cos t, \quad y = r(w)\sin t, \quad z = z(w),$$

where $w \in \langle w_1, w_2\rangle$, $t \in \langle 0, 2\pi)$. The parameters w_1 and w_2 can reach values $\pm\infty$.

The metric of the surface $\mathcal{S}_2$ has the following form

$$\mathrm{d}s^2 = a(w)\,\mathrm{d}w^2 + b(w)\,\mathrm{d}t^2, \tag{43}$$

where $a(w)$ and $b(w)$ are non-zero functions and where

$$a(w) = r'^2(w) + z'^2(w) \quad \text{and} \quad b(w) = r^2(w).$$

Let us suppose that the surface $\mathcal{S}_2$ maps geodesically onto a surface of revolution $\bar{\mathcal{S}}_2$, such that parallels and meridians on $\mathcal{S}_2$ map onto parallels and meridians on $\bar{\mathcal{S}}_2$. From this follows that the equations of the surface $\bar{\mathcal{S}}_2$ have the following form

$$x = \bar{r}(w)\cos t, \quad y = \bar{r}(w)\sin t, \quad z = \bar{z}(w),$$

where w is the same parameter as on the surface $\mathcal{S}_2$ and $t \in \langle 0, 2\pi)$.

The metric of the surface $\bar{\mathcal{S}}_2$ has the following form

$$\mathrm{d}\bar{s}^2 = \big(\bar{r}'^2(w) + \bar{z}'^2(w)\big)\,\mathrm{d}w^2 + \bar{r}^2(w)\,\mathrm{d}t^2.$$

It was proved by Mikeš [62], see also [70] (p. 302), that the surface $\mathcal{S}_2$ with the metric (43) maps geodesically onto the surface $\bar{\mathcal{S}}_2$ with the metric

$$\mathrm{d}\bar{s}^2 = \frac{p\,a(w)}{\big(1 + qb(w)\big)^2}\,\mathrm{d}w^2 + \frac{p\,b(w)}{1 + qb(w)}\,\mathrm{d}t^2,$$

where p and q are real parameters, t and w are coordinates.

The surface $\bar{\mathcal{S}}_2$ can be obtained from the surface $\mathcal{S}_2$ using these transformations:

$$\bar{r}(w) = \frac{r(w)}{\sqrt{1 + qr^2(w)}}\,, \bar{z}(w) = \int_{w_0}^{w} \sqrt{\frac{z'^2 + qr^2\left(r'^2 + z'^2\right)}{\left(1 + qr^2\right)^3}}\, \mathrm{d}\tilde{w}, \tag{44}$$

where q is a parameter. The coordinate w is the same as on the surface $\mathcal{S}_2$. The functions $\bar{r}$ and $\bar{z}$ mentioned above have to fulfil the condition of smoothness at the poles $w = w_1$ and $w = w_2$, where $\bar{r} = 0$, namely $\dfrac{\mathrm{d}\bar{r}}{\mathrm{d}w} = \pm 1$ and $\dfrac{\mathrm{d}\bar{z}}{\mathrm{d}w} = 0$.

The mapping is a non-trivial geodesic mapping if $q \neq 0$ and $b \not\equiv 0$, and the transformation is non-trivial if $b \not\equiv 0$. We restrict our further study to $q \geq 0$.

3.1.2 Geodesic mapping of a circular cylinder

Let $\mathcal{S}_2$ be a circular cylinder given by the parametric equations

$$x = k\cos t, \quad y = k\sin t, \quad z = w,$$

where k (> 0) is its radius. The metric of the surface $\mathcal{S}_2$ has the form $\mathrm{d}s^2 = \mathrm{d}w^2 + k^2\, \mathrm{d}t^2$.

Using relations (44), we can characterize the surface $\bar{\mathcal{S}}_2$ by the following equations

$$\bar{r}(w) = \frac{k}{\sqrt{1 + qk^2}} \quad \text{and} \quad \bar{z}(w) = \int_{w_1}^{w} \sqrt{\frac{1 + qk^2}{\left(1 + qk^2\right)^3}}\, \mathrm{d}\tilde{w} = \frac{1}{1 + qk^2}\,(w - w_1).$$

Evidently, $\bar{r}(w)$ is a constant, therefore surface $\bar{\mathcal{S}}_2$ is also a circular cylinder. Moreover, from the above follows, that the geodesic transformation is trivial.

3.1.3 Geodesic mapping of a circular cone

Let $\mathcal{S}_2$ be a circular cone given by the parametric equations

$$x = w\cos t, \quad y = w\sin t, \quad z = w,$$

Now let $w > 0$, thus we are considering the top part of the cone. For the value $w = 0$ we get the vertex of the cone which is a singular point.

The equations of a generalized circular cone can be written in the following form

$$x = aw\cos t, \quad y = aw\sin t, \quad z = cw,$$

where $a,\ c \in \mathbb{R}$. We choose $a = c = 1$. Any other circular cone can be obtained from the considered one by a certain affine transformation of $\mathcal{E}_3$.

The metric of the circular cone $\mathcal{S}_2$ has the form $\mathrm{d}s^2 = 2\,\mathrm{d}w^2 + w^2\,\mathrm{d}t^2$.

Using the transformation equations (44) we get

$$\bar{r}(w) = \frac{w}{\sqrt{1+qw^2}} \quad \text{and} \quad \bar{z}(w) = \int_0^w \sqrt{\frac{1+2q\tilde{w}^2}{(1+q\tilde{w}^2)^3}}\,\mathrm{d}\tilde{w}. \tag{45}$$

We can choose the lower bound of the integral as 0. For $q = 0$ is $\bar{\mathcal{S}}_2 \equiv \mathcal{S}_2$. If $q > 0$ then $\bar{r}(w) > 0$.

The metric of the surface $\bar{\mathcal{S}}_2$ has the following form

$$\mathrm{d}\bar{s}^2 = \frac{2}{(1+qw^2)^2}\,\mathrm{d}w^2 + \frac{w^2}{1+qw^2}\,\mathrm{d}t^2.$$

We consider the following limits of equations (45) for $w \to \infty$

$$\bar{r}_{\max} = \lim_{w\to\infty} \bar{r}(w) = \lim_{w\to\infty} \frac{w}{\sqrt{1+qw^2}} = \frac{1}{\sqrt{q}}$$

and

$$\bar{z}_{\max} = \lim_{w\to\infty} \bar{z}(w) = \lim_{w\to\infty} \int_0^w \sqrt{\frac{1+2q\tilde{w}^2}{(1+q\tilde{w}^2)^3}}\,\mathrm{d}\tilde{w} \le \lim_{w\to\infty} \int_0^w \frac{\sqrt{2}}{1+q\tilde{w}^2}\,\mathrm{d}\tilde{w} =$$

$$= \sqrt{\frac{2}{q}} \lim_{w\to\infty} \arctan\left(\sqrt{q}\,w\right) = \frac{\pi}{\sqrt{2q}}\,.$$

The transformation of the surface $\mathcal{S}_2$ onto $\bar{\mathcal{S}}_2$ is non-trivial. The circular conical surface deforms onto the surface, which is no longer a quadric. For this mapping all geodesics (except "parallels") map onto incomplete geodesics on the surface $\bar{\mathcal{S}}_2$.

Let $\bar{\mathcal{S}}_2$ be an image of the cone $\mathcal{S}_2$ in non-trivial geodesic deformation.

Proposition 1. *A circular conical surface admits a non-trivial global geodesic transformation in which it remains rotational. The surface $\bar{\mathcal{S}}_2$ is no longer a quadric surface.*

3.1.4 Geodesic mapping of a one-sheet hyperboloid of revolution

Let $\mathcal{S}_2$ be a one-sheet hyperboloid of revolution given by the parametric equations

$$x = \cosh w\, \cos t, \quad y = \cosh w\, \sin t, \quad z = \sinh w.$$

The metric of the surface $\mathcal{S}_2$ has the form

$$\mathrm{d}s^2 = (\sinh^2 w + \cosh^2 w)\, \mathrm{d}w^2 + \cosh^2 w\ \mathrm{d}t^2.$$

Using transformations (44) we get

$$\bar{r}(w) = \frac{\cosh w}{\sqrt{1 + q\cosh^2 w}} \tag{46}$$

and

$$\bar{z}(w) = \int_0^w \frac{\cosh \tilde{w}}{1 + q\cosh^2 \tilde{w}} \cdot \sqrt{\frac{1 + q\,(2\cosh^2 \tilde{w} - 1)}{1 + q\cosh^2 \tilde{w}}}\ \mathrm{d}\tilde{w}. \tag{47}$$

Notice that the functions $\bar{r}(w)$ and $\bar{z}(w)$ are monotonically increasing for $w \geq 0$ and $q > 0$. Let us consider the limit of (46) for $w \to \infty$

$$\bar{r}_{\max} = \lim_{w\to\infty} \frac{\cosh w}{\sqrt{1 + q\cosh^2 w}} = \frac{1}{\sqrt{q}}.$$

Instead of finding an explicit solution of the integral (47), let us consider

$$\bar{z}_{\max} = \lim_{w\to\infty} \int_0^w \frac{\cosh \tilde{w}}{1 + q\cosh^2 \tilde{w}} \cdot \sqrt{\frac{1 + q\,(2\cosh^2 \tilde{w} - 1)}{1 + q\cosh^2 \tilde{w}}}\ \mathrm{d}\tilde{w}.$$

The limit exists and we can estimate its value in this manner

$$\bar{z}_{\max} \leq \lim_{w\to\infty} \int_0^w \frac{\cosh \tilde{w}}{1 + q\cosh^2 \tilde{w}} \sqrt{2}\ \mathrm{d}\tilde{w} \leq \frac{1}{q} \lim_{w\to\infty} \int_0^w \frac{\cosh \tilde{w}}{\cosh^2 \tilde{w}}\ \mathrm{d}\tilde{w} =$$
$$= \frac{1}{q} \lim_{w\to\infty} \int_0^w \frac{1}{\cosh \tilde{w}}\ \mathrm{d}\tilde{w} = \frac{\pi}{2q}.$$

From this follows that

$$\bar{z}_{\max} \le \frac{\pi}{2q} \quad \text{and similarly} \quad \bar{z}_{\min} \ge -\frac{\pi}{2q}.$$

The geodesics on the one-sheet hyperboloid of revolution are complete. Evidently, the geodesics on the surface $\bar{\mathcal{S}}_2$ are incomplete (except "parallels").

Let $\bar{\mathcal{S}}_2$ be an image of the one-sheet hyperboloid $\mathcal{S}_2$ in non-trivial geodesic deformation.

Proposition 2. *A one-sheet hyperboloid of revolution admits a non-trivial geodesic transformation in which it remains rotational. The surface $\bar{\mathcal{S}}_2$ is no longer a quadric surface.*

3.1.5 Geodesic mapping of a two-sheet hyperboloid of revolution

Let $\mathcal{S}_2$ be a two-sheet hyperboloid of revolution given by the equations

$$x = \sinh w \cos t, \quad y = \sinh w \sin t, \quad z = \cosh w.$$

Note that the equations mentioned above describe only the top part of the hyperboloid. The metric of the hyperboloid has the following form

$$\mathrm{d}s^2 = (\cosh^2 w + \sinh^2 w)\,\mathrm{d}w^2 + \sinh^2 w\ \mathrm{d}t^2.$$

Using formulas (44), we obtain the transformation equations for the hyperboloid

$$\bar{r}(w) = \frac{\sinh w}{\sqrt{1 + q\sinh^2 w}}, \tag{48}$$

$$\bar{z}(w) = \int_{w_0}^{w} \frac{\sinh \tilde{w}}{1 + q\sinh^2 \tilde{w}} \cdot \sqrt{\frac{1 + q\,(2\sinh^2 \tilde{w} + 1)}{1 + q\sinh^2 \tilde{w}}}\ \mathrm{d}\tilde{w}. \tag{49}$$

For the pole $w_1 = 0$ where $\bar{r}(w_1) = 0$, the smoothness conditions are fulfilled, therefore $\dfrac{\mathrm{d}\bar{r}}{\mathrm{d}w}(w_1) = 1$ and $\dfrac{\mathrm{d}\bar{z}}{\mathrm{d}w}(w_1) = 0$.

For $q = 0$ formulas (48) and (49), when $w_0 = 0$, have the solution $\bar{r} = \sinh w$, $\bar{z} = \cosh w$. For $q \approx 0$ these solutions are $\bar{r} \approx \sinh w$ and $\bar{z} \approx \cosh w$, which can be verified by the integration of Taylor series.

Similarly as on the one-sheet hyperboloid of revolution, we study limits of (48) and (49) for $w \to \infty$

$$\bar{r}_{\max} = \lim_{w\to\infty} \frac{\sinh w}{\sqrt{1 + q \sinh^2 w}} = \frac{1}{\sqrt{q}}$$

and

$$\begin{aligned}\bar{z}_{\max} = \lim_{w\to\infty} \int_1^w \frac{\sinh \tilde{w}}{1 + q \sinh^2 \tilde{w}} \cdot \sqrt{\frac{1 + q\,(2\sinh^2 \tilde{w} + 1)}{1 + q \sinh^2 \tilde{w}}}\, \mathrm{d}\tilde{w} \leq \\ \leq \lim_{w\to\infty} \int_1^w \frac{\sqrt{2}\sinh \tilde{w}}{1 + q \sinh^2 \tilde{w}}\, \mathrm{d}\tilde{w} \leq \lim_{w\to\infty} \int_1^w \frac{1}{q \sinh \tilde{w}}\, \mathrm{d}\tilde{w} = \\ = -\frac{1}{q} \ln\left(\tanh\left(\frac{1}{2}\right)\right) \approx \frac{1}{q} \cdot 0,77194.\end{aligned}$$

Evidently, from the calculated limits, the two-sheet hyperboloid of revolution deforms to a surface of revolution which is bounded in space (as in example above). Note, that all geodesics on $\bar{\mathcal{S}}_2$ (except "parallels") are incomplete.

Let $\bar{\mathcal{S}}_2$ be an image of the hyperboloid $\mathcal{S}_2$ in non-trivial geodesic deformation.

Proposition 3. *A two-sheet hyperboloid of revolution admits a non-trivial geodesic mapping in which it remains rotational. The surface $\bar{\mathcal{S}}_2$ is no longer a quadric surface.*

3.1.6 Geodesic mapping of a paraboloid of revolution

Let $\mathcal{S}_2$ be a paraboloid of revolution given by the equations

$$x = w \cos t, \quad y = w \sin t, \quad z = w^2.$$

The metric of the paraboloid has the form $\mathrm{d}s^2 = (1 + 4w^2)\,\mathrm{d}w^2 + w^2\,\mathrm{d}t^2$.

The transformation equations for this paraboloid are

$$\bar{r}(w) = \frac{w}{\sqrt{1+qw^2}} \quad \text{and} \quad \bar{z}(w) = \int_0^w \frac{\tilde{w}}{1+q\tilde{w}^2} \cdot \sqrt{\frac{4+q(1+4\tilde{w}^2)}{1+q\tilde{w}^2}}\, \mathrm{d}\tilde{w}. \tag{50}$$

We verify if the smoothness conditions are satisfied for the pole $w = w_1$ where $\bar{r} = 0$. Evidently, for the pole $w_1 = 0$ applies $\frac{\mathrm{d}\bar{z}}{\mathrm{d}w}(w_1) = 0$ and further $\frac{\mathrm{d}\bar{r}}{\mathrm{d}w}(w_1) = 1$, thus both conditions are fulfilled.

As for the previous surfaces we find limits of (50) for $w \to \infty$

$$\bar{r}_{\max} = \lim_{w\to\infty} \frac{w}{\sqrt{1+qw^2}} = \frac{1}{\sqrt{q}},$$

$$\bar{z}_{\max} = \lim_{w\to\infty} \int_0^w \frac{\tilde{w}}{1+q\tilde{w}^2} \cdot \sqrt{\frac{4+q\,(1+4\tilde{w}^2)}{1+q\tilde{w}^2}}\, \mathrm{d}\tilde{w} \le \lim_{w\to\infty} \frac{2}{2q} \ln|1+qw^2| = +\infty.$$

From the calculated limits and transformation equations (50) it follows that the circular paraboloid $\mathcal{S}_2$ maps non-trivially geodesically onto the surface $\bar{\mathcal{S}}_2$ which is no longer a quadric surface.

Proposition 4. *A circular paraboloid admits a non-trivial geodesic deformation in which it remains rotational, but the obtained surface is no longer a quadric surface.*

The results of all the propositions can be summarized in form of a theorem [R3].

Theorem 8. *Quadric surface of revolution (except a circular cylinder) admit non-trivial geodesic mappings and deformations under which they remain surfaces of revolution. Surfaces (except circular cylinder and sphere) obtained in these geodesic deformations are no longer quadric surfaces.*

3.2 Geodesic mappings of spaces with affine connection

Further on, we present the results from the theory of geodesic mappings of generalized Ricci symmetric manifolds. Ricci simmetric manifolds are a natural generalization of the Ricci manifolds.

Let us suppose, that studied objects are continuous and smooth enough.

Let $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ be a diffeomorphism between spaces $\mathbb{A}_n = (M, \nabla)$ and $\bar{\mathbb{A}}_n = (\bar{M}, \bar{\nabla})$ with affine connections ∇ and $\bar{\nabla}$, where M and $\bar{M}$ are n-dimensional manifolds. We suppose that $M \equiv \bar{M}$ and therefore in coordinate neighbourhood (U, x) corresponding points $x \in M$ and $f(x) \in \bar{M}$ have "common" coordinates $(x^1, x^2, \dots, x^n)$.

We define a *deformation tensor of affine connections respective mapping* $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ in the following form $P = \bar{\nabla} - \nabla$, i.e. their components in common coordinate system x are

$$P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x), \tag{51}$$

where $\Gamma^h_{ij}(x)$ and $\bar{\Gamma}^h_{ij}(x)$ are components of connections ∇ and $\bar{\nabla}$, respectively.

A diffeomorphism $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ is called a *geodesic mapping* of $\mathbb{A}_n$ onto $\bar{\mathbb{A}}_n$ if f maps any geodesic in $\mathbb{A}_n$ onto a geodesic in $\bar{\mathbb{A}}_n$.

It is known [21, 22, 68, 70, 80, 90, 95], that diffeomorphism $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ is a geodesic mapping if and only if in common coordinate system $x = (x^1, x^2, \dots, x^n)$ the deformation tensor (51) has the following form

$$P^h_{ij}(x) = \psi_i(x)\delta^h_j + \psi_j(x)\delta^h_i, \tag{52}$$

where $\psi_i(x)$ are components of a covector ψ and δ^h_i is the Kronecker symbol. Geodesic mapping is *non-trivial* if $\psi_i(x) \not\equiv 0$.

Evidently, any space $\mathbb{A}_n$ admits a non-trivial geodesic mapping onto the other space $\bar{\mathbb{A}}_n$. On the contrary, the statement is not valid for geodesic mappings onto Riemannian spaces.

Particularly, there exist spaces with affine connections which do not admit non-trivial geodesic mappings onto (pseudo-) Riemannian spaces, see [27, 57, 64–68, 70, 76–78, 80, 90, 94, 95].

It was proved [8, 11] that the fundamental equations of geodesic mappings of spaces with an affine connection onto Riemannian spaces as well as fundamental equations of geodesic mappings of spaces with an affine connection onto Ricci symmetric spaces form a closed Cauchy-type system of equations in the covariant derivative. Furthermore, this system is linear in the case of geodesic mappings onto Riemannian spaces.

3.2.1 Geodesic mappings of spaces with affine connection onto generalized Ricci symmetric spaces

Geodesic mappings onto *generalized Ricci symmetric spaces* were studied in [12], see also [70, pp. 469–473]. Fundamental equations of these mappings were derived and presented in the form of a closed Cauchy-type system of equations in the covariant derivative. In the following text, we find a simpler form of fundamental equations of these geodesic mappings.

Let $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ be a geodesic mapping of space $\mathbb{A}_n$ with an affine connection onto generalized Ricci symmetric spaces $\bar{\mathbb{A}}_n$. We supposed that $x = (x^1, x^2, \ldots, x^n)$ is a common coordinate system respective mapping f.

Space $\bar{\mathbb{A}}_n$ is called a *generalized Ricci symmetric space* if Ricci tensor satisfies the following condition

$$\bar{R}_{ij|k} + \bar{R}_{kj|i} = 0, \tag{53}$$

where $\bar{R}_{ij}$ are components of Ricci tensor on $\bar{\mathbb{A}}_n$ and the symbol "$|$" denotes a covariant derivative on $\bar{\mathbb{A}}_n$.

Let us remark, that Riemannian space which is a generalized Ricci symmetric space is also *Ricci symmetric* ($\bar{R}_{ij|k} = 0$).

Because the covariant derivative of Riemannian tensor has the following form

$$\bar{R}^h_{ijk|m} = \frac{\partial \bar{R}^h_{ijk}}{\partial x^m} + \Gamma^h_{m\alpha}\bar{R}^\alpha_{ijk} - \bar{\Gamma}^\alpha_{mi}\bar{R}^h_{\alpha jk} - \bar{\Gamma}^\alpha_{mj}\bar{R}^h_{i\alpha k} - \bar{\Gamma}^\alpha_{mk}\bar{R}^h_{ij\alpha},$$

from condition (51) we obtain

$$\bar{R}^h_{ijk|m} = \bar{R}^h_{ijk,m} + P^h_{m\alpha}\bar{R}^\alpha_{ijk} - P^\alpha_{mi}\bar{R}^h_{\alpha jk} - P^\alpha_{mj}\bar{R}^h_{i\alpha k} - P^\alpha_{mk}\bar{R}^h_{ij\alpha}, \tag{54}$$

where " , " denotes a covariant derivative on $\mathbb{A}_n$, R^h_{ijk} and $\bar{R}^h_{ijk}$ are components of Riemannian tensors of $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectively.

After contracting (54) respective indices h and k, we get

$$\bar{R}_{ij|m} = \bar{R}_{ij,m} - P^\alpha_{mi}\bar{R}_{\alpha j} - P^\alpha_{mj}\bar{R}_{i\alpha} \tag{55}$$

and after symmetrization (55) respective indices i and m, we get

$$\bar{R}_{ij|m} + \bar{R}_{mj|i} = \bar{R}_{ij,m} + \bar{R}_{mj,i} - 2P^\alpha_{mi}\bar{R}_{\alpha j} - P^\alpha_{mj}\bar{R}_{i\alpha} - P^\alpha_{ij}\bar{R}_{m\alpha}. \tag{56}$$

Since space $\bar{\mathbb{A}}_n$ is generalized Ricci symmetric, then it satisfies equation (53). Therefore, from (56) follows

$$\bar{R}_{ij,m} + \bar{R}_{mj,i} = 2P^\alpha_{mi}\bar{R}_{\alpha j} + P^\alpha_{mj}\bar{R}_{i\alpha} + P^\alpha_{ij}\bar{R}_{m\alpha}. \tag{57}$$

The deformation tensor P^h_{ij} of affine connections has the form (52) and from the formula (57) it follows

$$\bar{R}_{ij,m} + \bar{R}_{mj,i} = 3\psi_m\bar{R}_{ij} + 3\psi_i\bar{R}_{mj} + \psi_j(\bar{R}_{im} + \bar{R}_{mi}). \tag{58}$$

It is known [95] and [70, p. 182], that between Riemannian tensors R^h_{ijk} and $\bar{R}^h_{ijk}$ of $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$ the following dependence holds

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^\alpha_{ik}P^h_{j\alpha} - P^\alpha_{ij}P^h_{k\alpha}. \tag{59}$$

From (52) follows

$$P^h_{ij,k} = \psi_{i,k}\delta^h_j + \psi_{j,k}\delta^h_i$$

and from formula (59), we obtain

$$\bar{R}^h_{ijk} = R^h_{ijk} - \delta^h_j \psi_{i,k} + \delta^h_k \psi_{i,j} - \delta^h_i \psi_{j,k} + \delta^h_i \psi_{k,j} + \delta^h_j \psi_i \psi_k - \delta^h_k \psi_i \psi_j.$$

We contract last formula respective indices h and k. Finally, we get

$$\bar{R}_{ij} = R_{ij} + n\psi_{i,j} - \psi_{j,i} + (1-n)\psi_i \psi_j \tag{60}$$

and after alternating (60), we obtain

$$\bar{R}_{[ij]} = R_{[ij]} + (n+1)\psi_{i,j} - (n+1)\psi_{j,i}, \tag{61}$$

where $[ij]$ denotes an alternation respective indices i and j.

From condition (61), we obtain

$$\psi_{i,j} - \psi_{j,i} = \frac{1}{n+1}\left(\bar{R}_{[ij]} - R_{[ij]}\right). \tag{62}$$

By analysing (60) and (62), we get the following equation

$$\psi_{i,j} = \frac{1}{n^2-1}\left[n\bar{R}_{ij} + \bar{R}_{ji} - (nR_{ij} + R_{ji})\right] + \psi_i \psi_j. \tag{63}$$

Now let us covariantly differentiate, respective x^k in space $\mathbb{A}_n$, the equation (58) and substitute (63), then we get

$$\bar{R}_{ij,mk} + \bar{R}_{mj,ik} = 3\psi_m \bar{R}_{ij,k} + 3\psi_i \bar{R}_{mj,k} + \psi_j(\bar{R}_{im,k} + \bar{R}_{mi,k}) + T_{ijmk}, \tag{64}$$

where

$$\begin{aligned} T_{ijmk} = 3&\left(\frac{1}{n^2-1}\left(n\bar{R}_{mk} + \bar{R}_{km} - (nR_{mk} + R_{km})\right) + \psi_m \psi_k\right)\bar{R}_{ij} + \\ &+3\left(\frac{1}{n^2-1}\left(n\bar{R}_{ik} + \bar{R}_{ki} - (nR_{ik} + R_{ki})\right) + \psi_i \psi_k\right)\bar{R}_{mj} + \\ &+\left(\frac{1}{n^2-1}\left(n\bar{R}_{jk} + \bar{R}_{kj} - (nR_{jk} + R_{kj})\right) + \psi_j \psi_k\right)\left(\bar{R}_{im} + \bar{R}_{mi}\right). \end{aligned}$$

We alternate condition (64) respective indices i and k:

$$\begin{aligned} \bar{R}_{ij,mk} - \bar{R}_{kj,mi} = &\bar{R}_{\alpha j} R^\alpha_{mki} + \bar{R}_{m\alpha} R^\alpha_{jki} + 3\psi_m \bar{R}_{ij,k} - 3\psi_m \bar{R}_{kj,i} + \\ &+3\psi_i \bar{R}_{mj,k} - 3\psi_k \bar{R}_{mj,i} + \psi_j(\bar{R}_{im,k} - \bar{R}_{km,i} + \bar{R}_{mi,k} - \bar{R}_{mk,i}) + \\ &+T_{ijmk} - T_{kjmi}. \end{aligned}$$

From Ricci identity and algebraic identy of Riemannian tensor, we obtain

$$\begin{aligned}\bar{R}_{ij,km} - \bar{R}_{kj,im} = 2\bar{R}_{\alpha j}R^{\alpha}_{mki} + \bar{R}_{i\alpha}R^{\alpha}_{jkm} + \bar{R}_{k\alpha}R^{\alpha}_{jmi} + \bar{R}_{m\alpha}R^{\alpha}_{jki} + \\ +3\psi_m\bar{R}_{ij,k} - 3\psi_m\bar{R}_{kj,i} + 3\psi_i\bar{R}_{mj,k} - 3\psi_k\bar{R}_{mj,i} + \\ +\psi_j(\bar{R}_{im,k} - \bar{R}_{km,i} + \bar{R}_{mi,k} - \bar{R}_{mk,i}) + T_{ijmk} - T_{kjmi}.\end{aligned} \tag{65}$$

In (65) we replace indices k and m, and to results add (64). Finally, we get

$$\begin{aligned}2\bar{R}_{ijm,k} = 2\bar{R}_{\alpha j}R^{\alpha}_{kim} + \bar{R}_{i\alpha}R^{\alpha}_{jmk} + \bar{R}_{m\alpha}R^{\alpha}_{jki} + \bar{R}_{k\alpha}R^{\alpha}_{jmi} + \\ +3\psi_k(\bar{R}_{ijm} - \bar{R}_{mji}) + 3\psi_i(\bar{R}_{kjm} + \bar{R}_{mjk}) + 3\psi_m(\bar{R}_{ijk} - \bar{R}_{kji}) + \\ +\psi_j(\bar{R}_{imk} + \bar{R}_{mik} + \bar{R}_{ikm} - \bar{R}_{mki} + \bar{R}_{kim} - \bar{R}_{kmi}) + \\ +T_{ijkm} - T_{mjki} + T_{ijmk},\end{aligned} \tag{66}$$

where $\bar{R}_{ijm} = \bar{R}_{ij,m}$.

Evidently, equations (63), (66) and

$$\bar{R}_{ij,m} = \bar{R}_{ijm} \tag{67}$$

on space $\mathbb{A}_n$ form a closed Cauchy-type system respective unknown functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, and $\bar{R}_{ijk}(x)$.

In conclusion, we formulate the following [R11].

Theorem 9. *A space $\mathbb{A}_n$ with an affine connection admits geodesic mapping onto generalized Ricci symmetric spaces $\bar{\mathbb{A}}_n$ if and only if on $\mathbb{A}_n$ exists the solution of a closed Cauchy-type system of equations in the covariant derivative* (63), (66), *and* (67) *respective unknown functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, and $\bar{R}_{ijk}(x)$.*

The general solution of this system depends on no more then

$$n + n^2 + \frac{1}{2}n^2(n-1) \equiv \frac{1}{2}n^2(n+1) + n$$

real parameters.

From the conditions (51) and (52) follows that space $\bar{\mathbb{A}}_n$ from the Theorem 9 has an equiaffine connection

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \psi_j + \delta^h_j \psi_i,$$

where ψ_i is a solution of the system mentioned above. Furthermore, from it follows that $\bar{R}_{ij}$ are components of the Ricci tensor on $\bar{\mathbb{A}}_n$.

3.2.2 Geodesic mappings of spaces with affine connection onto equiaffine generalized Ricci symmetric spaces

It is known [95], [70, pp. 85], that equiaffine spaces are defined by the symmetry of Ricci tensor. We verify, that the following lemma holds.

Lemma 1. *Equiaffine generalized Ricci symmetric space is Ricci symmetric.*

Proof. Let $\bar{\mathbb{A}}_n$ be a generalized Ricci symmetric space for which the condition (53) holds. Since $\bar{\mathbb{A}}_n$ is equiaffine then

$$\bar{R}_{ij} = \bar{R}_{ji}. \tag{68}$$

First, after differentiating (68) we obtain

$$\bar{R}_{ij|k} = \bar{R}_{ji|k}. \tag{69}$$

Then, from the properties (53) and (69), it follows

$$\bar{R}_{ij|k} \overset{(53)}{=} -\bar{R}_{ik|j} \overset{(69)}{=} -\bar{R}_{ki|j} \overset{(53)}{=} \bar{R}_{kj|i} \overset{(69)}{=}$$
$$\overset{(69)}{=} \quad \bar{R}_{jk|i} \overset{(53)}{=} -\bar{R}_{ji|k} \overset{(69)}{=} -\bar{R}_{ij|k}.$$

Finally, we compare the first and the last article and verify that

$$\bar{R}_{ij|k} = 0,$$

therefore, $\bar{\mathbb{A}}_n$ is Ricci symmetric. □

Geodesic mappings onto Ricci symmetric manifolds were studied in [8]. In our case, the equations of such mapping would be simpler because $\bar{\mathbb{A}}_n$ is equiaffine. This system of equations, using [8], has the form:

$$\bar{R}_{ij,m} = 2\psi_m \bar{R}_{ij} + \psi_i \bar{R}_{mj} + \psi_j \bar{R}_{im}, \tag{70}$$

$$\psi_{i,j} = \frac{1}{n^2-1}\left[(n+1)\bar{R}_{ij} - (nR_{ij} + R_{ji})\right] + \psi_i\psi_j. \tag{71}$$

It is evident that the equations (70) and (71) in the given manifold represent a closed Cauchy-type system with respect to unknown functions $\bar{R}_{ij}(x)$ and $\psi_i(x)$.

Theorem 10 ([R11])**.** *A manifold $\mathbb{A}_n$ with affine connection admits a geodesic mapping onto an equiaffine Ricci symmetric manifold $\bar{\mathbb{A}}_n$ if and only if in $\mathbb{A}_n$ exists a solution of a closed Cauchy-type equations in the covariant derivative* (70) *and* (71) *with respect to unknown functions $\bar{R}_{ij}(x)$ $(= \bar{R}_{ji}(x))$ and $\psi_i(x)$.*

The general solution of a closed Cauchy-type system of equations (70) and (71) depends on no more than $\frac{1}{2}n(n+1) + n \equiv \frac{1}{2}n(n+3)$ independent real parameters.

From the conditions (51) and (52) follows that space $\bar{\mathbb{A}}_n$ from the Theorem 10 has an equiaffine connection

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i\psi_j + \delta^h_j\psi_i,$$

where ψ_i is solution of the system mentioned above. Furthermore, from it follows that $\bar{R}_{ij}$ are components of the Ricci tensor on $\bar{\mathbb{A}}_n$.

3.2.3 Geodesic mappings of spaces with affine connections onto Ricci-2-symmetric spaces

The paper [72] extends the results about geodesic mappings of (pseudo-) Riemannian spaces to geodesic mappings of equiaffine spaces with affine connections onto (pseudo-)

Riemannian spaces. Mikeš [57] studied geodesic mappings of generalized symmetric and recurrent (pseudo-) Riemannian spaces.

The paper [8] proves that the main equations of geodesic mappings of spaces with affine connections onto Ricci-symmetric spaces are equivalent to some Cauchy-type system of differential equations in covariant derivatives. In the following part, main equations for geodesic mappings of spaces with affine connections onto Ricci-2-symmetric spaces are obtained as a closed Cauchy-type system of differential equations in covariant derivatives. We determine the number of essential parameters on which the solution of the system depends. The results are extended for geodesic mappings of spaces with affine connections onto Ricci-m-symmetric spaces.

We suppose, throughout this chapter, that all geometric objects under considerations are continuous and sufficiently smooth.

A space $\bar{\mathbb{A}}_n$ with affine connection (Riemannian space $\bar{\mathbb{V}}_n$) is called Ricci-m-symmetric if its Ricci tensor $\bar{R}_{ij}$ satisfies the condition

$$\bar{R}_{ij|k_1k_2\dots k_m} = 0, \tag{72}$$

where the symbol "$|$" denotes a covariant derivative with respect to the connection of the space $\bar{\mathbb{A}}_n$ (cf. e.g. [37–39], [70, p. 338]). In particular, for the case of Ricci-2-symmetric spaces (72) is written as follows:

$$\bar{R}_{ij|km} = 0. \tag{73}$$

We also introduced the tensor $\bar{R}_{ijk}$ defined by (67)

$$\bar{R}_{ij,k} = \bar{R}_{ijk} \ . \tag{74}$$

According to [70, 80, 84, 95], a necessary and sufficient condition for the mapping f of a space $\mathbb{A}_n$ onto a space $\bar{\mathbb{A}}_n$ to be geodesic is that in the common coordinate system $x^1, x^2, \dots, x^n$ the *deformation tensor* $P^h_{ij}(x)$ of the mapping f

$$P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x), \tag{75}$$

has to satisfy the condition

$$P_{ij}^h(x) = \psi_i(x)\delta_j^h + \psi_j(x)\delta_i^h. \tag{76}$$

The symbols $\Gamma_{ij}^h(x)$ and $\bar{\Gamma}_{ij}^h(x)$ are components of affine connections of the spaces $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectively, $\psi_i(x)$ are components of a covariant vector.

A geodesic mapping is called *non-trivial* if $\psi_i(x) \neq 0$. It is obvious, that any space $\mathbb{A}_n$ with an affine connection admits a non-trivial geodesic mapping onto some space $\bar{\mathbb{A}}_n$ with an affine connection. It is not difficult to construct an example of a non-trivial geodesic mapping. Let $\mathbb{A}_n$ be a space with affine connection Γ_{ij}^h. By determining an arbitrary vector field ψ_i in $\mathbb{A}_n$, we construct the non-trivial geodesic mapping of the space $\mathbb{A}_n$ onto a space $\bar{\mathbb{A}}_n$ with the affine connection $\bar{\Gamma}_{ij}^h$. Using (75) and (76) we can calculate the components of $\bar{\Gamma}_{ij}^h$. However, in general, the same does not apply for geodesic mappings of Riemannian spaces onto Riemannian spaces. In particular, there are Riemannian spaces that do not admit non-trivial geodesic mappings onto Riemannian spaces.

Let us consider a geodesic mapping of a space $\mathbb{A}_n$ with an affine connection onto a Ricci-2-symmetric space $\bar{\mathbb{A}}_n$. Yet, in general, spaces with affine connection, and in particular Ricci-2-symmetric spaces, are not (pseudo-) Riemannian spaces.

Suppose that the spaces $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$ are referred to a coordinate system common to the mapping.

It is known [70, 95] that a relationship between the Riemann tensors R_{ijk}^h and $\bar{R}_{ijk}^h$ of the spaces $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectively, is presented by the formulas:

$$\bar{R}_{ijk}^h = R_{ijk}^h + P_{ik,j}^h - P_{ij,k}^h + P_{ik}^\alpha P_{\alpha j}^h - P_{ij}^\alpha P_{\alpha k}^h\ . \tag{77}$$

Taking into account that a deformation tensor P_{ij}^h of the connections is defined by (76), it follows from (77) that

$$\bar{R}_{ijk}^h = R_{ijk}^h - \delta_j^h\psi_{i,k} + \delta_k^h\psi_{i,j} - \delta_i^h\psi_{j,k} + \delta_i^h\psi_{k,j} + \delta_j^h\psi_i\psi_k - \delta_k^h\psi_i\psi_j\ . \tag{78}$$

Contracting the equations (78) for h and k, we get

$$\bar{R}_{ij} = R_{ij} + n\psi_{i,j} - \psi_{j,i} + (1-n)\psi_i\psi_j \ . \tag{79}$$

Alternating (79) with respect to the indices i and j, we obtain

$$\bar{R}_{[ij]} = R_{[ij]} + (n+1)\psi_{i,j} - (n+1)\psi_{j,i} \ . \tag{80}$$

Here, by the brackets $[ij]$ we denote an operation called antisymmetrization (or alternation) without division with respect to the indices i and j. Taking account of (79), from (80) it follows

$$\psi_{i,j} = \frac{1}{n^2-1}\left[n\bar{R}_{ij} + \bar{R}_{ji} - (nR_{ij} + R_{ji})\right] + \psi_i\psi_j \ . \tag{81}$$

It was proved [R13] that $\bar{R}_{ij|k} = \bar{R}_{ij,k} - P^{\alpha}_{ki}\bar{R}_{\alpha j} - P^{\alpha}_{kj}\bar{R}_{i\alpha}$. Using this relation and considering that the deformation tensor is defined by (76), we find

$$\bar{R}_{ij|k} = \bar{R}_{ij,k} - 2\psi_k\bar{R}_{ij} - \psi_i\bar{R}_{kj} - \psi_j\bar{R}_{ik} \ . \tag{82}$$

Differentiating (82) with respect to x^m in the space $\mathbb{A}_n$, we obtain

$$(\bar{R}_{ij|k})_{,m} = \bar{R}_{ij,km} - 2\psi_{k,m}\bar{R}_{ij} - 2\psi_k\bar{R}_{ij,m} - \psi_{i,m}\bar{R}_{kj} - \psi_i\bar{R}_{kj,m} - \psi_{j,m}\bar{R}_{ik} - \psi_j\bar{R}_{ik,m} \ . \tag{83}$$

Taking into account $\left(\bar{R}_{ij|k}\right)_{,m} = \bar{R}_{ij|km} + P^{\alpha}_{mi}\bar{R}_{\alpha j|k} + P^{\alpha}_{mj}\bar{R}_{i\alpha|k} + P^{\alpha}_{mk}\bar{R}_{ij|\alpha}$ and (76), from (83) it follows

$$\begin{aligned}\bar{R}_{ij|km} = \bar{R}_{ij,km} - 2\psi_{k,m}\bar{R}_{ij} - 2\psi_k\bar{R}_{ij,m} - \psi_{i,m}\bar{R}_{kj} - \psi_i\bar{R}_{kj,m} - \psi_{j,m}\bar{R}_{ik} - \\ - \psi_j\bar{R}_{ik,m} - \psi_i\bar{R}_{mj|k} - 3\psi_m\bar{R}_{ij|k} - \psi_j\bar{R}_{im|k} - \psi_k\bar{R}_{ij|m} \ .\end{aligned} \tag{84}$$

Suppose that the space $\bar{\mathbb{A}}_n$ is Ricci-2-symmetric. Then, taking account of (74), (81) and (82), we have from (84)

$$\begin{aligned}\bar{R}_{ijk,m} = 2\rho_{km}\bar{R}_{ij} + 2\psi_k\bar{R}_{ijm} + \rho_{im}\bar{R}_{kj} + \psi_i\bar{R}_{kjm} + \rho_{jm}\bar{R}_{ik} + \psi_j\bar{R}_{ikm} + \\ + \psi_i\rho_{mjk} + 3\psi_m\rho_{ijk} + \psi_j\rho_{imk} + \psi_k\rho_{ijm} \ ,\end{aligned} \tag{85}$$

where

$$\begin{aligned}\rho_{ij} =& \frac{1}{n^2-1}\left(n\bar{R}_{ij} + \bar{R}_{ji} - \left(R_{ij} + R_{ji}\right)\right) + \psi_i\psi_j,\\ \rho_{ijk} =& \bar{R}_{ijk} - 2\psi_k\bar{R}_{ij} - \psi_i\bar{R}_{kj} - \psi_j\bar{R}_{ik}.\end{aligned}$$

It is obvious that in the space $\bar{\mathbb{A}}_n$ the equations (74), (85), and (81) form a closed Cauchy-type system of differential equations of in covariant derivatives with respect to functions $\psi_i(x)$, $\bar{R}_{ij}(x)$ and $\bar{R}_{ijk}(x)$. Hence we proved the following theorem [R13].

Theorem 11. *Let $\mathbb{A}_n$ be a space with an affine connection and let $\bar{\mathbb{A}}_n$ be a Ricci-*2*-symmetric space. Then $\mathbb{A}_n$ admits geodesic mapping onto $\bar{\mathbb{A}}_n$ if and only if the closed Cauchy-type system of differential equations in covariant derivatives* (74)*,* (85) *and* (81) *has a solution with respect to functions $\psi_i(x)$, $\bar{R}_{ij}(x)$ and $\bar{R}_{ijk}(x)$.*

The general solution of the closed Cauchy-type system of differential equations in covariant derivatives (74), (85), and (81) depends on no more than $\frac{1}{2}n \cdot (n+1)^2 + n$ essential parameters.

Similarly, as in the case of conformal mappings, see [R14], the main equations for geodesic mappings of spaces with affine connections onto a Ricci-m-symmetric space could be obtained in the form of a closed Cauchy-type system of equations in covariant derivatives.

3.3 On canonical almost geodesic mappings of type $\pi_2(e)$

This section develops some new ideas in the theory of almost geodesic mappings of spaces with the affine connection.

Special almost geodesic mappings π_2 are mappings of type $\pi_2(e)$, which are related to e-structure F ($F^2 = e \cdot Id,\ e = \pm 1, 0$), defined on the manifold, see [95]. The next part is devoted to the study of the conditions guaranteeing that the Riemann tensor is invariant with respect to the canonical almost geodesic mappings of type $\pi_2(e)$. In addition, we study canonical almost geodesic mappings of type $\pi_2(e)$ of spaces with affine connections onto symmetric spaces. The main equations for the mappings are derived as a closed, mixed system of Cauchy-type PDE's in covariant derivatives.

The investigations use local coordinates. We assume that all functions under consideration are sufficiently differentiable.

3.3.1 Basic definitions of almost geodesic mappings of spaces with affine connections

Let us recall the basic definition, formulas, and theorems of the theory presented in [69, 70, 95, 96], [R1]. Consider a space $\mathbb{A}_n$ with an affine connection ∇ without torsion. The space is referred to with coordinates $x = (x^1, x^2, \dots, x^n)$.

Let us recall that curve ℓ: $x = x(t)$ in the space $\mathbb{A}_n$ is *geodesic* if its tangent vector $\lambda(t) = dx(t)/dt$ satisfies the equations $\nabla_t \lambda = 0$, see pp. 21–21.

A curve ℓ in the space $\mathbb{A}_n$ is an *almost geodesic* when its tangent vector λ satisfies the equations

$$\nabla_t \nabla_t \lambda = a(t)\lambda + b(t)\nabla_t \lambda,$$

where $a(t)$ and $b(t)$ are certain functions of t.

A diffeomorphism $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ is called *almost geodesic* if any geodesic curve of $\mathbb{A}_n$ is mapped under f onto an almost geodesic curve in $\bar{\mathbb{A}}_n$.

Suppose that a space $\mathbb{A}_n$ with the affine connection ∇ admits a mapping f onto a space $\bar{\mathbb{A}}_n$ with an affine connection $\bar{\nabla}$, and the spaces are referred to with the common coordinate system $x = (x^1, x^2, \dots, x^n)$.

The tensor $P = \bar{\nabla} - \nabla$ is called a *deformation tensor* of the connections ∇ and $\bar{\nabla}$ with respect to the mapping f. In common coordinates x, components of P have the form:

$$P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x),$$

where $\Gamma^h_{ij}(x)$ and $\bar{\Gamma}^h_{ij}(x)$ are components of affine connections of the spaces $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectively.

According to [95], a necessary and sufficient condition for the mapping $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ to be almost geodesic is that the deformation tensor $P^h_{ij}(x)$ of the mapping f must satisfy the condition

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma = a \cdot P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + b \cdot \lambda^h,$$

where λ^h is an arbitrary vector and a and b are certain functions of variables $x^1, x^2, \dots, x^n$ and $\lambda^1, \lambda^2, \dots, \lambda^n$. The tensor A^h_{ijk} is defined as

$$A^h_{ijk} = P^h_{ij,k} + P^\alpha_{ij} P^h_{\alpha k}.$$

We denote by comma "," a covariant derivative with respect to the connection of the space $\mathbb{A}_n$.

Almost geodesic mappings of spaces with affine connection were introduced in [95] by N. S. Sinyukov. He distinguished three kinds of almost geodesic mappings, namely, π_1, π_2, and π_3, characterized by following conditions for the deformation tensor P:

$$\pi_1: \quad A^h_{(ijk)} = \delta^h_{(i} a_{jk)} + b_{(i} P^h_{jk)} \ ,$$

$$\pi_2: \quad P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)}, \qquad F^h_{(i,j)} + F^h_\alpha F^\alpha_{(i}\varphi_{j)} = \delta^h_{(i}\mu_{j)} + F^h_{(i}\rho_{j)} \ ,$$

$$\pi_3: \quad P^h_{ij} = \delta^h_{(i}\psi_{j)} + \theta^h a_{ij}, \qquad \theta^h_{,i} = \rho \cdot \delta^h_i + \theta^h a_i,$$

where δ_i^h is the Kronecker symbol, the round parentheses of indices denote an operation called symmetrization without division, and $F_i^h, \theta^h, a_{ij},\ a_i, \psi_i, \varphi_i, \mu_i, \rho_i, \rho$ are tensors.

The types of almost geodesic mappings π_1, π_2, π_3 can intersect. The problem of completeness of classification was resolved; Berezovski and Mikeš [9] proved that, for $n > 5$, other types of almost geodesic mappings, except π_1, π_2, and π_3, do not exist.

3.3.2 Almost geodesic mappings $\pi_2(e),\ e = \pm 1, 0$

A mapping π_2 satisfies the *mutuality condition* if the inverse mapping is also an almost geodesic of type π_2 and corresponds to the same affinor $F_i^h(x)$.

The mappings π_2 satisfying the mutuality condition will be denoted as $\pi_2(e)$, where $e = \pm 1, 0$, see [95], and is characterized by the following equations:

$$P_{ij}^h = \delta_{(i}^h \psi_{j)} + F_{(i}^h \varphi_{j)}, \tag{86}$$

$$F_{(i,j)}^h = F_{(i}^h \mu_{j)} - \delta_{(i}^h F_{j)}^\alpha \mu_\alpha \quad \text{and} \quad F_\alpha^h F_i^\alpha = e\delta_i^h. \tag{87}$$

As it was proved in [75], in case $e = \pm 1$, the basic equations of the mappings $\pi_2(e)$ can be written as Equation (86), and

$$F_{i,j}^h = F_{ij}^h, \quad F_{ij,k}^h = \overset{6}{\Theta}{}_{ijk}^h, \quad \mu_{i,j} = \mu_{ij}, \quad \mu_{ij,k} = \overset{7}{\Theta}_{ijk}, \tag{88}$$

$$F_{(ij)}^h = F_{(i}^h \mu_{j)} - \delta_{(i}^h F_{j)}^\alpha \mu_\alpha, \qquad F_\alpha^h F_i^\alpha = e\delta_i^h, \qquad \mu_{(ij)} = \overset{5}{\Theta}_{ij}, \tag{89}$$

where

$$\overset{1}{\Theta}{}_{ijk}^h \equiv \overset{2}{\Theta}{}_{ijk}^h + \overset{2}{\Theta}{}_{kji}^h - \overset{2}{\Theta}{}_{jki}^h + 2F_\alpha^h R_{kji}^\alpha - F_i^\alpha R_{\alpha jk}^h + F_j^\alpha R_{\alpha ik}^h + F_k^\alpha R_{\alpha ij}^h,$$
$$\overset{2}{\Theta}{}_{ijk}^h \equiv \mu_{(i} F_{j)k}^h - \delta_{(i}^h F_{j)k}^\alpha \mu_\alpha,$$
$$\overset{3}{\Theta}{}_{ijk}^h \equiv \overset{2}{\Theta}{}_{ijk}^h - \overset{2}{\Theta}{}_{kji}^h + F_j^\alpha R_{\alpha ik}^h - F_\alpha^h R_{jik}^\alpha,$$

$$
\begin{aligned}
&\overset{4}{\Theta}_{jk} \equiv F^\alpha_\beta \overset{1}{\Theta}{}^\beta_{\alpha jk} + 2F^\alpha_{\beta j} F^\beta_{\alpha k},\\
&\overset{5}{\Theta}_{jk} \equiv \frac{1}{(n-1-F^\alpha_\alpha)^2-1}\big((n-1-F^\alpha_\alpha)\overset{4}{\Theta}_{ij} + \overset{4}{\Theta}_{\alpha\beta} F^\alpha_i F^\beta_j\big),\\
&\overset{6}{\Theta}{}^h_{ijk} \equiv \frac{1}{2}\big(F^h_i \mu_{(jk)} + F^h_j \mu_{[ik]} + F^h_k \mu_{[ij]} - \delta^h_i m_{(jk)} - \delta^h_j m_{[ik]} - \delta^h_k m_{[ij]} + \overset{1}{\Theta}{}^h_{ikj}\big),\\
&\overset{7}{\Theta}_{ijk} \equiv \mu_\alpha R^\alpha_{kji} + \frac{1}{2}\big(\overset{5}{\Theta}_{ij,k} + \overset{5}{\Theta}_{ik,j} - \overset{5}{\Theta}_{jk,i}\big), \quad m_{ij} \equiv F^\alpha_i \mu_{\alpha j},
\end{aligned}
$$

F^h_i, F^h_{ij}, μ_i, μ_{ij} are unknown functions, and R^h_{ijk} is the Riemann tensor of the space $\bar{\mathbb{A}}_n$. We denote by the brackets $[ik]$ an operation called antisymmetrization (or alternation) without division with respect to the indices i and k.

Obviously, right-hand sides of Equation (88) depend on unknown functions F^h_i, F^h_{ij}, μ_i, μ_{ij} and on the components Γ^h_{ij} of the space $\mathbb{A}_n$. Then, Equations (88) and (89) form a closed, mixed system of Cauchy-type PDE's with respect to functions F^h_i, F^h_{ij}, μ_i, μ_{ij}. The general solution of the system, Equations (88) and (89), depends on no more than $\frac{1}{2}\, n(n+1)^2$ essential parameters. In addition, the mapping $\pi_2(e)$ depends on unknown functions ψ_i, φ_j (see Equation (86)).

3.3.3 Canonical almost geodesic mappings $\pi_2(e)$ $(e = \pm 1)$ preserving the Riemann tensor

An almost geodesic mapping π_2 for which $\psi_i = 0$ is called *canonical*. It is known that any almost geodesic mapping π_2 can be written as the composition of a canonical almost geodesic mapping and a geodesic mapping. The latter may be referred to as a trivial almost geodesic mapping.

Hence, a canonical almost geodesic mapping $\pi_2(e)$ $(e = \pm 1)$ is determined by the equation

$$
P^h_{ij} = F^h_i \varphi_j + F^h_j \varphi_i, \tag{90}
$$

and also by Equations (88) and (89).

In [7] it was proved that Riemann tensor is preserved by the diffeomorphism if and only if the tensor A^h_{ijk} satisfies the conditions

$$A^h_{ijk} = A^h_{ikj}. \tag{91}$$

If the deformation tensor P^h_{ij} is expressed by Equation (90), then for $\pi_2(e)$ $(e = \pm 1)$ taking account of (87), (88), and (89), we get

$$A^h_{ijk} = \varphi_{i,k}F^h_j + \varphi_{j,k}F^h_i + \varphi_i\big(F^h_{jk} + \varphi_\alpha F^\alpha_j F^h_k + e\delta^h_j\varphi_k\big) + \varphi_j\big(F^h_{ik} + \varphi_\alpha F^\alpha_i F^h_k + e\delta^h_i\varphi_k\big).$$

Now, we require that A^h_{ijk} satisfies (91). Hence,

$$\varphi_{i,k}F^h_j - \varphi_{i,j}F^h_k + \varphi_{j,k}F^h_i - \varphi_{k,j}F^h_i = B^h_{ijk}, \tag{92}$$

where

$$B^h_{ijk} = \varphi_k\big(F^h_{ij} + \varphi_\alpha F^\alpha_i F^h_j + e\delta^h_i\varphi_j\big) - \varphi_j\big(F^h_{ik} + \varphi_\alpha F^\alpha_i F^h_k + e\delta^h_i\varphi_k\big) +$$
$$+\varphi_i\big(F^h_{kj} + \varphi_\alpha F^\alpha_k F^h_j + e\delta^h_k\varphi_j - F^h_{jk} - \varphi_\alpha F^\alpha_j F^h_k - e\delta^h_j\varphi_k\big).$$

Let us multiply (92) by F^j_h and contract for indices h and j. Hence, we have

$$n\varphi_{i,k} - \varphi_{k,i} = eB^\alpha_{i\beta k}F^\beta_\alpha. \tag{93}$$

Symmetrizing (93) in i and k, we obtain

$$\varphi_{i,k} + \varphi_{k,i} = \frac{e}{n-1}F^\beta_\alpha\big(B^\alpha_{i\beta k} + B^\alpha_{k\beta i}\big). \tag{94}$$

Equations (93) and (94) can be written as

$$\varphi_{i,k} = \frac{e}{n+1}F^\beta_\alpha\Big(B^\alpha_{i\beta k} + \frac{1}{n-1}\big(B^\alpha_{i\beta k} + B^\alpha_{k\beta i}\big)\Big). \tag{95}$$

Hence, we get the theorem, [R14].

Theorem 12. *Let $\mathbb{A}_n$ be a space with affine connection preserving the Riemanian tensor and let $\bar{\mathbb{A}}_n$ be a space with affine connection. Then $\mathbb{A}_n$ admits almost geodesic mappings of type $\pi_2(e)$ if and only if the mixed system of Cauchy-type differential equations in covariant derivatives* (88) *and* (95) *has a solution with respect to unknown functions F^h_i, F^h_{ij}, μ_i, μ_{ij}, φ_i which must satisfy the algebraic conditions* (89).

The general solution of the system (88), (89), and (95) depends on no more than $\frac{1}{2}\, n(n+1)^2 + n$ essential parameters.

3.3.4 Canonical almost geodesic mappings $\pi_2(e)$ of spaces with affine connection onto symmetric spaces

A space $\mathbb{A}_n$ with an affine connection is called *(locally) symmetric* if its Riemann tensor is absolutely parallel. Symmetric spaces were introduced by P. A. Shirokov [93] and É. Cartan in 1932 [14], see also S. Helgason [24, 25]. Therefore, the symmetric spaces $\bar{\mathbb{A}}_n$ are characterized by

$$\bar{R}^h_{ijk|m} = 0, \tag{96}$$

where $\bar{R}^h_{ijk}$ is the Riemann tensor of the space $\bar{\mathbb{A}}_n$. By the symbol "$\,|\,$" we denote covariant derivative with respect to the connection of the space $\bar{\mathbb{A}}_n$.

Let us consider the canonical almost geodesic mappings of type $\pi_2(e)$ $(e = \pm 1)$ of spaces $\mathbb{A}_n$ with an affine connection onto symmetric spaces $\bar{\mathbb{A}}_n$, which are determined by Equations (90), (88), and (89). Suppose that the spaces are referred to the common coordinate system $x^1, x^2, \dots, x^n$.

Since

$$\bar{R}^h_{ijk|m} = \frac{\partial \bar{R}^h_{ijk}}{\partial x^m} + \bar{\Gamma}^h_{m\alpha}\bar{R}^\alpha_{ijk} - \bar{\Gamma}^\alpha_{mi}\bar{R}^h_{\alpha jk} - \bar{\Gamma}^\alpha_{mj}\bar{R}^h_{i\alpha k} - \bar{\Gamma}^\alpha_{mk}\bar{R}^h_{ij\alpha},$$

then taking account of (87), we can obtain

$$\bar{R}^h_{ijk|m} = \bar{R}^h_{ijk,m} + P^h_{m\alpha}\bar{R}^\alpha_{ijk} - P^\alpha_{mi}\bar{R}^h_{\alpha jk} - P^\alpha_{mj}\bar{R}^h_{i\alpha k} - P^\alpha_{mk}\bar{R}^h_{ij\alpha}. \tag{97}$$

In what follows, we understand that the space $\bar{\mathbb{A}}_n$ is symmetric. Taking account of (90) and (96), we have from (97) that

$$\bar{R}^h_{ijk,m} = \varphi_{(i}F^\alpha_{m)}\bar{R}^h_{\alpha jk} + \varphi_{(j}F^\alpha_{m)}\bar{R}^h_{i\alpha k} + \varphi_{(k}F^\alpha_{m)}\bar{R}^h_{ij\alpha} - \varphi_{(m}F^h_{\alpha)}\bar{R}^\alpha_{ijk}. \tag{98}$$

It is known [95] that the Riemann tensors of the spaces $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$ are related to each other by the equations

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^k_{ik,j} - P^h_{ij,k} + P^\alpha_{ik} P^h_{\alpha j} - P^\alpha_{ij} P^h_{\alpha k}. \tag{99}$$

Because the deformation tensor of the mapping $P^h_{ij}(x)$ is represented by Equation (90), it follows from (99) that

$$\varphi_{i,j} F^h_k + \varphi_{k,j} F^h_i - \varphi_{i,k} F^h_j - \varphi_{j,k} F^h_i = D^h_{ijk}, \tag{100}$$

where

$$D^h_{ijk} = \bar{R}^h_{ijk} - R^h_{ijk} - \varphi_i\big(F^h_{kj} + \varphi_\alpha F^\alpha_k F^h_j + e\delta^h_k \varphi_j - F^h_{jk} - \varphi_\alpha F^\alpha_j F^h_k - e\delta^h_j \varphi_k\big) + \\ + \varphi_k\big(F^h_{ij} + \varphi_\alpha F^\alpha_i F^h_j\big) - \varphi_j\big(F^h_{ik} + \varphi_\alpha F^\alpha_i F^h_k\big).$$

Let us multiply (100) by F^k_h and contract for h and k. Hence, we have

$$n\varphi_{i,j} - \varphi_{j,i} = e D^\alpha_{i\beta j} F^\beta_\alpha. \tag{101}$$

Symmetrizing (101) in i and j, we obtain

$$\varphi_{i,j} + \varphi_{j,i} = \frac{e}{n-1} F^\beta_\alpha \big(D^\alpha_{i\beta j} + D^\alpha_{j\beta i}\big). \tag{102}$$

Equations (101) and (102) can be written as

$$\varphi_{i,j} = \frac{e}{n+1} F^\beta_\alpha \big(D^\alpha_{i\beta j} + \frac{1}{n-1}\big(D^\alpha_{i\beta j} + D^\alpha_{j\beta i}\big)\big). \tag{103}$$

Obviously, Equations (88), (98), and (103) form a closed, mixed system of Cauchy-type PDE's with respect to functions F^h_i, F^h_{ij}, μ_i, μ_{ij}, $\bar{R}^h_{ijk}$, φ_i, and the functions F^h_i, F^h_{ij}, μ_i, μ_{ij} must satisfy the algebraic conditions (89). The algebraic conditions for the functions $\bar{R}^h_{ijk}$ are the Bianci identity

$$\bar{R}^h_{ijk} + \bar{R}^h_{ikj} = 0, \quad \text{and} \quad \bar{R}^h_{ijk} + \bar{R}^h_{jki} + \bar{R}^h_{kij} = 0. \tag{104}$$

Hence, we have proved the theorem, [R14].

Theorem 13. *Let* $\mathbb{A}_n$ *be a space with an affine connection and* $\bar{\mathbb{A}}_n$ *a symmetric space. Then,* $\mathbb{A}_n$ *admits an almost geodesic mappings of type* $\pi_2(e)$ $(e = \pm 1)$ *onto* $\bar{\mathbb{A}}_n$ *if and only if the mixed Cauchy-type system of differential equations in covariant derivatives* (88), (98)*, and* (103) *has a solution with respect to unknown functions* F_i^h, F_{ij}^h, μ_i, μ_{ij}, $\bar{R}_{ijk}^h$, φ_i *which must satisfy the algebraic conditions* (89) *and* (104)*.*

It is obvious that the general solution of the mixed system of Cauchy-type PDE's depends on no more than

$$\frac{1}{3}\,n^2(n^2-1)+\frac{1}{2}\,n(n+1)^2+n$$

essential parameters.

Conclusion

This dissertation is devoted to the study of geodesics and their mappings, i.e., rotary, geodesic, and almost geodesic mappings. These mappings belong to a larger group of so-called special diffeomorphisms.

The First Chapter has presented geodesic bifurcations, particularly their existence in certain spaces. In future research, it might be possible to consider even more general spaces that would admit these bifurcations.

The Second Chapter has been dedicated to rotary mappings and their properties. The method that determines the existence of the rotary mappings of certain spaces has been constructed. It has been shown that there exist Riemannian spaces that are not isometric to surfaces of revolution and admit rotary mappings. This statement is in contradiction with already known results. Therefore, future research can be focused on corrections of already published papers that were derived from the invalid statement.

The Third Chapter has developed the theory of geodesic and almost geodesic mappings. Certain special spaces have been considered, and the main equations of these mappings have been derived. Future work could consider more generalized spaces with special properties.

The results of this dissertation could be applied in the differential geometry of manifolds with structures and their mappings. They could also find an application in theoretical mechanics and thermodynamics. Results presented in the dissertation were included in monograph J. Mikeš et al. *Differential geometry of special mappings.*

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Geodesics and their mappings

Geodetické křivky a jejich zobrazení

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Contents

1 Introduction

Geometric structures, their mappings, and transformations on smooth manifolds represent the crucial part of contemporary differential geometry. Mainly, modern differential geometry focuses on mappings and transformations which preserve certain properties of geometric objects, known as properties that are invariant under these mappings and transformations.

As stated below, geodesics have great importance in differential geometry and its applications. The history of geodesics goes back to the 18th century to the work of J. Bernoulli and L. Euler. Next, geodesic mappings are known from the works of E. Beltrami and T. Levi-Civita in 19th century. The study of geodesic mappings was followed by a study of almost geodesic and rotary mappings dated back to the 20th century.

Geodesic, almost geodesic, and rotary mappings are special diffeomorphisms that map any geodesic of the first space onto geodesic, almost geodesic, and isoperimetric extremal of rotation, respectively. These topics are studied in detail in the monograph [R1].

The dissertation is dedicated to the topics mentioned in the previous paragraph. The main subject of this thesis is a study of geodesic bifurcations, rotary, geodesic, and almost geodesic mappings of (pseudo-) Riemannian spaces and spaces with affine connections.

Geodesics were first mentioned by Johann I. Bernoulli in his letter to Leibniz in 1697. Details are described in monograph [R1, pp. 119–121]. There are many textbooks devoted to the theory of geodesics, but geodesics also appear in many papers and special monographs about modern Differential Geometry, i.e. [1, 5, 7, 8, 9, 10, 17, 18, 25, 31, 33, 34, 35, 40, 43, 49, 54].

The First Chapter of the thesis deals with a question about the existence of geodesics. More precisely, there is an example of geodesic bifurcation, which can be described as a situation where two or more geodesics pass through a given point in a given direction, [R1, pp. 44–46], [R2, R5, R12], see also [49, 1].

Rotary mappings were introduced by Leiko in [20]; these are mappings that take any geodesic of one two-dimensional Riemannian space onto the isoperimetric extremal of rotation of another two-dimensional Riemannian space. Leiko and others published many papers on rotary mappings and related problems. In particular, they pointed out the importance of rotary mappings for physical applications [19, 21, 22, 24]. Leiko [23] also studied infinitesimal rotary transformations, for other examples of infinitesimal transformations see [12, 28, 32, 50, 51, 52]. Mikeš, Sochor, Stepanova [30] found new equations of isoperimetric extremals of rotation and Chudá, Mikeš, Sochor [6] also presented a more generalized definition of rotary mappings.

The Second Chapter of the thesis is dedicated to a further study of rotary mappings. For example, we have constructed a counterexample to known results and found conditions of the existence of the rotary mappings, [R1, pp. 137–148], [R4, R6, R7, R9, R10].

The Third Chapter includes new results in the theory of geodesic and almost geodesic mappings. We found the equations of geodesic and almost geodesic mappings between special manifolds, [R3, R11, R13, R14]. Further, we briefly present the theory of geodesic and almost geodesic mappings. There are many different approaches to this theory, as well as many results.

Geodesics and their generalizations are essential, especially in geometric structures studies. Let us mention E. Beltrami and his contribution to non-Euclidean geometry in 1865. Mappings that preserve geodesics were also studied during Beltrami's time, later known as *geodesic mappings*. T. Levi-Civita [25] provided some interesting applications of geodesic mappings in terms of dynamic processes modelling in mechanics. With this work, he contributed to the basis of geodesic mappings in tensor form.

T. Levi-Civita, T. Thomas, H. Weyl significantly contributed to the theory of general properties and dependencies of geodesic mappings. T. Levi-Civita found the equations describing geodesic mappings of Riemannian spaces and also found the metric form of Riemannian spaces that admit geodesic mappings. T. Thomas and H. Weyl found invariant objects of geodesic mappings. It is a fact that any space with affine connection is locally projectively equivalent to some equiaffine space. These "standard" results are stated and further developed in monographs by L. P. Eisenhart [8, 9].

N. S. Sinyukov [43] derived fundamental equations of geodesic mappings of Riemannian spaces, in a linear form, and thereby found conditions for the existence of the geodesic mapping. He also proved that the main equations for geodesic mappings of (pseudo-) Riemannian spaces are equivalent to some linear Cauchy-type system of differential equations in covariant derivatives. J. Mikeš and V. E. Berezovskii dealt with the same problem for geodesic mapping of spaces with an equiaffine connection onto Riemannian spaces. This result holds even for spaces with a general affine connection.

Following the study of geodesic mappings, the term *degree of mobility* of Riemannian manifolds with respect to geodesic mappings was defined. It is a number of real parameters on which a solution of a Cauchy-type system of partial differential equations depends. Here, the existence of the solution of the system yields the existence of geodesic mapping. In [16, 26], the authors found a connection between the degree of mobility with respect to geodesic mappings and degrees of isometric, homothetic, and projective transformations.

The geodesic mappings theory focuses on geodesic mappings of special spaces, namely spaces with constant curvature, Einstein, equidistant, symmetric, semi-symmetric, recurrent spaces, and their generalizations, see [2], [R1]. Geodesic deformations were studied by M. L. Gavrilčenko, J. Mikeš, and others. Finally, let us mention a study of geodesic mappings of spaces with affine connection, Finsler, and even more general spaces, see [26, 28, 31, 37, 39, 41, 43]. These mappings are described in detail in [R1].

Many monographs are dedicated to generalizations of geodesic mappings, such as almost geodesic mappings, holomorphically-projective mappings, F-planar mappings, conformally-projective mappings, and also their transformations and deformations.

The class of almost geodesic mappings is a natural generalization of the class of geodesic mappings. N. S. Sinyukov [43] introduced mapping, which maps any geodesic onto almost geodesic and entitled it *almost geodesic mapping*. He also established three types of almost geodesic mappings and denoted them π_1, π_2, and π_3. Later, V. Berezovskii and J. Mikeš [4, 27] specified this classification and proved that no other type than those three mentioned above exists.

In 1962, A. Z. Petrov [36] studied quasi-geodesic mappings and showed that they could be used to simulate physical processes and electromagnetic fields. Comparable results are presented in the paper of C. L. Bejan and O. Kowalski [3]. The mappings $\pi_2(e)$ are similar to those mentioned above. All these spaces are connected with some affinor structure F, which can be interpreted as a force field.

The theory of almost geodesic mappings was developed by V. S. Shadnyi [42] and many others [37, 38, 39, 45, 46, 53]. Almost geodesic mappings of symmetric spaces were studied by V. S. Sobchuk [44].

Nowadays, the theories of geodesics, geodesic mappings, and their generalizations were developed in many directions, i.e. in works [13, 14, 15, 47, 48].

2 The aim of the thesis

This Ph.D. thesis aims to study the existence of geodesic bifurcations. It also further develops results in the theory of rotary mappings and transformations. Other subjects of study are geodesic and almost geodesic mappings.

The main aims are:

- a detailed study of geodesic bifurcations and certain problems related to this topic;
- a detailed study of rotary mappings and transformations of (pseudo-) Riemannian spaces and the analysis of their equations;
- a study of geodesic and almost geodesic mappings of some special spaces.

3 Methods

We use classical tensor calculus for Riemannian and pseudo-Riemannian manifolds and manifolds with affine connection, in local form, as well as in global form.

4 Results

I. Geodesic bifurcations

[R1, pp. 44–46, 186–187], [R2, R5, R12]

– A surface of revolution that admits geodesic bifurcations was found.

– Based on the previous result, the bifurcation of closed geodesics was constructed.

– We constructed n-dimensional Riemannian and Kähler product spaces that admit local geodesic bifurcations and also bifurcations of closed geodesic.

II. Rotary mappings and transformations

[R1, pp. 437–448], [R4, R6, R7, R9, R10]

– We proved that there exist spaces which are not isometric to surfaces of revolution and concurrently admit rotary mappings.

– We performed an in-depth analysis of rotary vector field equations and obtained fundamental equations in the Cauchy-type form.

– We proved that any surface of revolution with differentiable Gaussian curvature admits rotary mappings.

– Lastly, we studied infinitesimal rotary transformations.

III. Geodesic and almost geodesic mappings

[R3, R11, R13, R14]

- We proved that under geodesic transformation, quadric surfaces of revolution map onto surfaces of revolution, which are no longer quadric surfaces. Furthermore, these surfaces are bounded in space.

- Additionally, we dealt with geodesic mappings of spaces with an affine connection onto generalized Ricci symmetric spaces. We obtained equations of these mappings in the form of a Cauchy-type system of equations in covariant derivatives. We specified the number of parameters for this system.

- We considered geodesic mappings of Riemannian spaces onto Ricci-2-symmetric Riemannian spaces. The main equations of the mappings were obtained as a closed Cauchy-type system of differential equations in covariant derivatives. We estimated the number of essential parameters on which the solution depends.

- We found conditions that must be fulfilled for the canonical almost geodesic mapping of $\pi_2(e)$ type to preserve Riemannian tensor. We derived the main equations of canonical almost geodesic mapping of $\pi_2(e)$ type in the form of a Cauchy-type system of partial differential equations (PDE's). The number of real parameters was estimated.

5 Overview of the Ph.D. thesis

The Ph.D. thesis consists of introduction, three chapters, conclusion and references.

I. Geodesic bifurcation

The **First Chapter** of the dissertation is dedicated to geodesics. The question about the existence of geodesic is studied in detail. It is proved that there exist geodesic bifurcations, that are described as situations when two or more geodesics pass through the given point in the given direction, [R2].

A different definition of bifurcations can be found in [49], where geodesics connect two points but do not have the same direction at any of them. Bifurcation theory is presented in [1].

Besides other examples in thesis, the following theorem is proved.

Theorem 1. *Let $\mathcal{S}_2$ be a surface of revolution given by the equations:*

$$x = r(u)\cos v, \quad y = r(u)\sin v, \quad z = z(u),$$

where v is a parameter from $(-\pi, \pi)$ and $u \in I \subset \mathbb{R}$, where $I = \langle u_1, u_2 \rangle$. Geodesic bifurcations exist on the surface of revolution $\mathcal{S}_2$ for $\alpha \in (0.5, 1)$.

These properties are usually omitted in the standard textbooks.

This result is used to construct the surface of revolution, where bifurcations of closed geodesic exist, [R5]. Next, there are constructed Riemannian and Kähler product spaces, where these bifurcations also exist. The last example shows that it is also possible to construct the pseudo-Riemannian spaces with bifurcations of isotropic geodesics, i.e., geodesics that have vanishing length, [R12].

II. Rotary mappings and transformations

The **Second Chapter** is devoted to rotary mappings of spaces with affine connection and Riemannian spaces.

In work [20] Leiko introduced the term *rotary mappings* between two-dimensional Riemannian spaces. Chudá, Mikeš and Sochor [6] later generalized this definition for the manifolds with affine connection.

Definition 1. A diffeomorfism f of two-dimensional manifold $\bar{\mathbb{A}}_2$ onto two-dimensional (pseudo-) Riemannian manifold $\mathbb{V}_2$ is called *rotary mapping* if any geodesic in $\bar{\mathbb{A}}_2$ is mapped onto isoperimetric extremal of rotation in $\mathbb{V}_2$.

Chudá, Mikeš and Sochor [6] formulated necessary and sufficient condition for the existence of the rotary mapping of the space with an affine connection $\bar{\mathbb{A}}_2$ onto the Riemannian space $\mathbb{V}_2$. This condition is the existence of a special torse-forming vector field θ (in $\mathbb{V}_2$) which satisfies

$$\nabla_X \theta = \theta \cdot (\Theta(X) + \nabla_X K/K) + \nu \cdot X \qquad (1)$$

for any tangent vector X, where ∇ is the Levi-Civita connection on $\mathbb{V}_2$, K is the Gaussian curvature, ν is a function, the form Θ is defined as $\Theta(X) = g(\theta, X)$, and g is a metric of the space $\mathbb{V}_2$.

The following theorem is proved.

Theorem 2. *There exists a Riemannian space $\mathbb{V}_2$ which is not isometric with a surface of revolution and where the vector field satisfying equations* (1) *exists.*

Moreover, Leiko [20] claimed that if Riemannian space admits rotary mapping then it is isometric to surface of revolution. This statement is not valid and the following theorem holds, [R6].

Theorem 3. *Let $\mathbb{A}_2$ be a space with affine connection, $\mathbb{V}_2$ any (pseudo-) Riemannian space, and let f be a rotary mapping of $\mathbb{A}_2$ onto $\mathbb{V}_2$. Then, the local structure of the metric of $\mathbb{V}_2$ has the form $ds^2 = \left(dx^1\right)^2 + f(x^1, x^2) \cdot \left(dx^2\right)^2$ where the function f satisfies conditions*

$$F' = -\frac{1}{2}\,F^2 + \frac{1}{\varkappa}\,F - 2 \cdot \frac{\varkappa'}{\varkappa^2},$$
$$f = c(x^2) \cdot \exp\left(\int F\,dx^1\right),$$

where $F = \partial_1 f/f$.

It is obvious, that the function $\exp\left(\int F\,dx^1\right)$ depends on both variables x^1, x^2. Therefore, the space $\mathbb{V}_2$ is not generally isometric to a surface of revolution.

In addition, there were found necessary condition for the existence of the vector fields described above, [R7].

Theorem 4. *A two-dimensional (pseudo-) Riemannian manifold $\mathbb{V}_2$ admits rotary vector field θ if and only if the following closed Cauchy-type system of PDE's in covariant derivatives has a solution with respect to functions $\theta_i(x)$ and $\nu(x)$:*

$$\theta_{i,j} = \theta_i(\theta_j + \partial_j K/K) + \nu g_{ij},$$
$$\nu_{,i} = \nu(\theta_i - \partial_i K/K) - K\theta_i - \theta_\alpha\theta_\beta g^{\alpha\beta}\partial_i K/K + \theta_i g^{\alpha\beta}\theta_\alpha\partial_\beta K/K.$$

I was proved [20], that only special surfaces of revolution admit rotary mappings. This result is used in many Leiko's papers. In the thesis, it is proved that rotary vector fields exist in any Riemannian space which is isometric with surface of revolution, thus the following holds, [R4, R10].

Theorem 5. *Let $\mathcal{S}_2$ be a surface of revolution and $\bar{\mathbb{A}}_2$ space with an affine connection. If Gaussian curvature K of the space $\mathcal{S}_2$ is differentiable then there exist a rotary mapping of $\bar{\mathbb{A}}_2$ onto $\mathcal{S}_2$.*

At the end of the Chapter, infinitesimal rotary transformations are defined and the following is proved, [R10].

Theorem 6. *A differential operator* $X = \xi^\alpha(x)\partial_\alpha$ $(\partial_\alpha = \partial/\partial x^\alpha)$ *determines an infinitesimal rotary transformation of (pseudo-) Riemannian space* $\mathbb{V}_2$ *if and only if* X *satisfies*

$$\mathcal{L}_\xi \Gamma^h_{ij} = \delta^h_{(i}\psi_{j)} + \theta^h g_{ij}, \qquad \theta^h_{,i} = \theta^h(\theta_i + K_i/K) + \nu\delta^h_i,$$

where ψ_i *is a covector,* δ^h_i *is the Kronecker delta,* θ^h *is a vector field,* g *is a metric tensor,* K $(\neq 0)$ *is the Gaussian curvature, and* $\mathcal{L}_\xi$ *is the Lie derivative with respect to* ξ.

These equations are in a simpler form, than equations presented by Leiko [23].

III. Geodesic and almost geodesic mappings

The **Third Chapter** is aimed to study geodesic and almost geodesic mappings of special spaces.

First, geodesic mappings of surfaces of revolution are studied in detail. These mappings are generalizations of results presented by I. Hinterleitner [11]. The following theorem is proved, [R3].

Theorem 7. *Quadric surface of revolution (except a circular cylinder) admit non-trivial geodesic mappings and deformations under which they remain surfaces of revolution. Surfaces (except circular cylinder and sphere) obtained in these geodesic deformations are no longer quadric surfaces.*

Next, the main equations of geodesic mappings of spaces with affine connection onto Ricci symmetric spaces and Ricci-2-symmetric spaces are derived in a form of a closed Cauchy-type system of equations in covariant derivatives. Similarly, the equations of geodesic mappings of spaces with affine connection onto equiaffine Ricci symmetric spaces are derived, [R11, R13].

Theorem 8. *A manifold $\mathbb{A}_n$ with affine connection admits a geodesic mapping onto an equiaffine Ricci symmetric manifold $\bar{\mathbb{A}}_n$ if and only if in $\mathbb{A}_n$ exists a solution of a closed Cauchy-type equations in the covariant derivative*

$$\bar{R}_{ij,m} = 2\psi_m \bar{R}_{ij} + \psi_i \bar{R}_{mj} + \psi_j \bar{R}_{im},$$
$$\psi_{i,j} = \frac{1}{n^2-1}\big[(n+1)\bar{R}_{ij} - (nR_{ij} + R_{ji})\big] + \psi_i\psi_j$$

with respect to unknown functions $\bar{R}_{ij}(x)$ $(=\bar{R}_{ji}(x))$ and $\psi_i(x)$.

The results mentioned above are generalizations of geodesic mappings of special spaces, namely constant curvature, Einstein, equidistant, symmetric, semi-symmetric, recurrent spaces, see [3, 8, 18, 25, 28, 35, 43, 54].

Finally, similar procedure is done for the almost geodesic mapping of type $\pi_2(e)$. Thus, main equations of geodesic mappings of special spaces are derived and two more theorems are formulated, [R14]. Again, these results are generalizations of almost geodesic mappings [29, 37, 38, 39, 42, 44, 45, 46, 53].

6 Summary

This Ph.D. thesis is devoted to the study of specific problems related to the theories of geodesics, rotary, geodesic, and almost geodesic mappings of manifolds with metric and affine structures.

The First Chapter deals with bifurcations of geodesics, i.e., with questions about the existence of two or more geodesics passing through the same point in the same direction. The original result is proof of the existence of surfaces of revolution that admit these bifurcations. In response to this result, surfaces where bifurcation of closed geodesic exists were constructed. Furthermore, (pseudo-) Riemannian and Kähler product spaces that contain these bifurcations are also constructed.

The Second Chapter is dedicated to rotary mappings and transformations. The Riemannian space that admits rotary mapping and at the same time is not isometric with a surface of revolution is found. This is a counterexample to known results. Another original result is a description of a method that determines whether the rotary vector field exists in (pseudo-) Riemannian space. The existence of this vector field yields the existence of rotary mapping onto (pseudo-) Riemannian spaces. The chapter is completed with a detailed study of rotary transformation.

The Third Chapter includes both geodesic and almost geodesic mappings. The geodesic transformations of quadric surfaces of revolution are studied in detail. Finally, the main equations of geodesic and almost geodesic mappings between some special manifolds are found in the form of a Cauchy-type system of partial differential equations.

Results presented in the Ph.D. thesis were included, especially in monograph J. Mikeš, L. Rýparová et al. Differential geometry of special mappings, Palacký University, Olomouc, 2019.

7 Anotace

Disertační práce je zaměřena na studium některých problémů spjatých s teoriemi geodetických křivek, rotačních, geodetických a téměř geodetických zobrazení variet s metrickými nebo afinními strukturami.

První kapitola disertační práce se zabývá geodetickými bifurkacemi, tj. otázkami existence více geodetik procházejících daným bodem ve stejném směru. Původním výsledkem je důkaz existence rotačních ploch, které připouštějí geodetické bifurkace. V návaznosti na získané výsledky jsou zkonsturovány plochy, na kterých existují bifurkace uzavřených geodetik. Dále jsou zkonstruovány (pseudo-) Riemannovy a Kählerovy produktové prostory, v nichž existují geodetické bifurkace.

Druhá kapitola je věnována rotačním zobrazením a transformacím. Jsou zde nalezeny Riemannovy prostory, které připouštějí rotační zobrazení a zároveň nejsou izometrické rotačním plochám. Toto je protipříklad k doposud známým výsledkům. Dalším výsledkem je popis metody, pomocí které lze určit, zda v (pseudo-) Riemannově prostoru existují rotační vektorová pole. Z existence zmiňovaných vektorových polí pak vyplývá i existence rotačních zobrazení na některé (pseudo-) Riemannovy prostory. Kapitola je ukončena studiem rotačních transformací.

Třetí kapitola pojednává o geodetických a téměř geodetických zobrazeních. Studují se zde geodetická zobrazení rotačních kvadrik. Jsou odvozeny rovnice geodetických a téměř geodetických zobrazení mezi speciálními varietami ve tvaru uzavřeného systému parciálních diferenciálních rovnic Cauchyho typu.

Výsledky předložené v této práci byly zařazeny zejména do monografie J. Mikeš, L. Rýparová et al. Differential geometry of special mappings, Palacký University, Olomouc, 2019.

8 AUTHOR's PUBLICATIONS related to the Ph.D. thesis

[R1] Mikeš J., Rýparová L. et al.: *Differential geometry of special mappings*, Olomouc: Palacký University, 2019, 676 p.
ISBN 978-80-244-5535-8, DOI 10.5507/prf.19.24455358

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