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DIPLOMOVÁ PRÁCE

Properties of solutions to the equations describing flow of fluids

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Prohlášení

Prohlašuji, že jsem vytvořila tuto diplomovou práci samostatně za vedení RNDr. Rostislava Vodáka, Ph.D. a že jsem v seznamu použité literatury uvedla všechny zdroje použité při zpracování práce.

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Poděkování

Ráda bych na tomto místě poděkovala vedoucímu diplomové práce RNDr. Rostislavu Vodákovi, Ph.D. za obětavou spolupráci i za čas, který mi věnoval při konzultacích.

Contents

Basic notation

$\mathbb{N}$	set of all integers
$\mathbb{R}$	set of all real numbers
$[a,b]$	closed interval in $\mathbb{R}$
(a,b)	open interval in $\mathbb{R}$
$\mathbb{R}^N$, $N \in \mathbb{N}$	real N-dimensional Euclidian space
$\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$	subset of $\mathbb{R}^N$
$\partial\Omega$	boundary of Ω
$\boldsymbol{u} \cdot \boldsymbol{v}$	scalar product of vectors
$\mathsf{A} : \mathsf{B}$	scalar product of matrices
$\boldsymbol{u} \otimes \boldsymbol{v}$	tensor product
$C^n(\Omega)$, $n \in \mathbb{N}$	set of all functions with countinuous derivatives up to the order n
$C^\infty(\Omega)$	set of all functions with countinuous derivatives of any order
$C_0^n(\Omega)$, $n \in \mathbb{N}$	set of all functions with countinuous derivatives up to the order n with compact support in Ω
$C_0^\infty(\Omega)$	set of all functions with countinuous derivatives of any order with compact support in Ω
$\mathcal{D}(\Omega)$	$C_0^\infty(\Omega)$ with the topology of locally uniform convergence
$\Vert \cdot \Vert_X$	norm in X
X^*	dual space of X
$\vert\Omega\vert$	Lebesgue measure of Ω
χ_Ω	characteristic function of Ω
$\partial_i u$	derivative of u according to i-th space variable
$\partial_t u$	derivative of u according to time variable
∇u	gradient of u
$\mathrm{D}\boldsymbol{u}$	symetric part of $\nabla \boldsymbol{u}$
$\hookrightarrow$	imbedding
$\hookrightarrow\hookrightarrow$	compact imbedding

Introduction

The aim of the diploma Thesis is to study the behaviour of the variational solutions to Navier-Stokes equations describing viscous compressible isothermal fluids with nonlinear stress tensors (for the formulation of the problem see Section 2) in a sequence of domains $\{\Omega_n\}_{n=1}^{\infty}$, which converges to a domain Ω. This convergence of domains is defined by the Sobolev-Orlicz capacity. We prove that the solutions converge to a solution of the respective Navier-Stokes equations in Ω. This problem was first studied in [1] for barotropic fluids.

The result obtained in the Thesis can be applied to generalization of the existence result proved in [4], [5] and [9], where $C^{2+\mu}$-regularity of the boundary of the domain was required. After the convergence of the sequence Ω_n, the existence result covers all Ω having $C^{0,1}$-regularity of its boundary. Moreover, the results of the Thesis provide mathematical apparatus for shape optimization.

The thesis is organized as follows: In Section 1, a corollary to of Alaouglu theorem is presented. Then Young functions and their properties are introduced. Next some theory of Orlicz and Sobolev-Orlicz spaces is explained, especially the spaces generated by fast growing functions. Sobolev-Orlicz capacity is also defined and the considered convergence of the domains introduced. In Section 2 the problem formulation and the main result of the Thesis is presented. Some auxiliary lemmas and their proofs are post poned to Section 3. These assertions will be used in the last section. In Section 4, apriori estimates are derived, which enable us to pass to weakly convergent subsequences. The last section uses the results of Sections 1, 3 and 4 to prove the main result.

1 Preliminaries

1.1 Some notes from functional analysis

Definition 1.1. Let X be a normed linear space and X^* its dual space. A sequence $\{u_n\}_{n=1}^{\infty} \subset X$ is said to *converge weakly* to $u \in X$ (denoted by $u_n \rightharpoonup u$), if $\varphi(u_n) \to \varphi(u)$ for any $\varphi \in X^*$.

Definition 1.2. Let X be a normed linear space. A sequence $\{\varphi_n\}_{n=1}^{\infty} \subset X^*$ is said to *converge weakly-$*$* to φ (denoted by $\varphi_n \overset{*}{\rightharpoonup} \varphi$), if $\varphi_n(u) \to \varphi(u)$ for any $u \in X$.

Theorem 1.3. *Let X be a separable normed linear space. Then every bounded sequence $\{\varphi_n\}_{n=1}^{\infty}$ in X^* contains a weak-$*$ convergent subsequence.*
Proof: For a sketch of the proof see [3, page 270].

1.2 Young functions and function spaces

Definition 1.4. We say that Φ is a *Young function* if

$$\Phi(z) = \int_0^z \varphi(y)\,\mathrm{d}y, \quad z \geq 0,$$

where φ is a real-valued function defined in $[0, \infty)$ such that

(i) $\varphi(0) = 0$,

(ii) $\varphi(y) > 0$ for $y > 0$,

(iii) φ is right continuous at any point $y \geq 0$,

(iv) φ is nondecreasing in $(0, \infty]$,

(v) $\lim\limits_{y \to \infty} \varphi(y) = \infty$.

Lemma 1.5. *Every Young function Φ is continuous, nonnegative, strictly increasing and convex in $[0, \infty)$. Moreover,*

$$\Phi(0) = 0, \quad \lim_{z \to \infty} \Phi(z) = \infty,$$

$$\lim_{z\to 0^+}\frac{\Phi(z)}{z}=0,\quad \lim_{z\to\infty}\frac{\Phi(z)}{z}=\infty,$$

$$\Phi(\alpha z)\le \alpha\Phi(z),\quad \alpha\in[0,1], z\ge 0,$$

$$\Phi(\beta z)\ge \beta\Phi(z),\quad \beta>1, z\ge 0.$$

Proof: See [2, page 129].

Definition 1.6. Let Φ be a Young function generated by the function φ, i.e.

$$\Phi(z)=\int_0^z \varphi(y)\,\mathrm{d}y.$$

If we denote

$$\psi(y)=\sup_{\varphi(z)\le y} z,\quad z\ge 0,$$

then the function

$$\Psi(z):=\int_0^z \psi(y)\,\mathrm{d}y$$

is called the *complementary function* to Φ.

Remark 1.7. If Φ is a Young function then its complementary function Ψ is a Young function as well.

Remark 1.8. If Ψ is complementary to Φ then Φ is complementary to Ψ. We can also call Φ, Ψ a pair of complementary Young functions.

Theorem 1.9 (Young's inequality)**.** *Let Φ, Ψ be a pair of complementary Young functions. Then for all $u, v\in[0,\infty)$ we have*

$$uv\le\Phi(u)+\Psi(v).$$

The equality occurs if and only if

$$v=\varphi(u)\quad \textit{or}\quad u=\psi(v).$$

Proof: For a sketch of the proof see [2, page 65].

Definition 1.10. A Young function Φ is said to satisfy the Δ_2-*condition* if there exist $k > 0$ and $z_0 \geq 0$ such that

$$\Phi(2z) \leq k\Phi(z), \quad \forall z \geq z_0.$$

If $z_0 = 0$, we call the Δ_2-condition *global*.

Lemma 1.11. *A Young function Φ satisfies the Δ_2-condition if and only if there exist $k > 0$ and $z_0 \geq 0$ such that*

$$\Psi(z) \leq \frac{1}{2k}\Psi(kz), \quad \forall z \geq z_0,$$

where Ψ is the complementary function to Φ.

Proof: See [2, page 139].

Definition 1.12. Let Φ_1, Φ_2 be two Young functions. If there exist positive constants k and z_0 such that

$$\Phi_1(z) \leq \Phi_2(kz) \quad \text{for } z \geq z_0,$$

we write down this fact as

$$\Phi_1 \prec \Phi_2.$$

If

$$\Phi_1 \prec \Phi_2 \quad \text{and} \quad \Phi_2 \prec \Phi_1$$

we say that Φ_1 and Φ_2 are *equivalent*.

Definition 1.13. Let Φ_1, Φ_2 be two Young functions. If

$$\lim_{z\to\infty} \frac{\Phi_1(z)}{\Phi_2(\lambda z)} = 0$$

for all $\lambda > 0$, we denote it by

$$\Phi_1 \prec\prec \Phi_2.$$

Theorem 1.14 (Jensen's inequality)**.** *Let Φ be convex in $\mathbb{R}$.*

- *Let $u_1, \ldots, u_n \in \mathbb{R}$ and $\alpha_1, \ldots, \alpha_n$ be positive numbers. Then*

$$\Phi\left(\frac{\alpha_1 u_1 + \cdots + \alpha_n u_n}{\alpha_1 + \cdots + \alpha_n}\right) \leq \frac{\alpha_1 \Phi(u_1) + \cdots + \alpha_n \Phi(u_n)}{\alpha_1 + \cdots + \alpha_n}. \tag{1.1}$$

- *Let $\alpha(\boldsymbol{x})$ be defined and positive almost everywhere in $\Omega \subset \mathbb{R}^N$. Then*

$$\Phi\left(\frac{\int_\Omega u(\boldsymbol{x})\alpha(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}{\int_\Omega \alpha(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}\right) \leq \frac{\int_\Omega \Phi(u(\boldsymbol{x}))\alpha(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}{\int_\Omega \alpha(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}} \tag{1.2}$$

  for every nonnegative function u provided all the integrals are finite.

Proof: See [2, page 133].

Definition 1.15. Let $\Omega \subset \mathbb{R}^N$ be an open set and let Φ be a nonnegative function defined in $[0, \infty)$. The set of all Lebesgue-measurable functions u defined almost everywhere in Ω such that

$$\rho(u; \Phi) := \int_\Omega \Phi(|u(\boldsymbol{x})|)\,\mathrm{d}\boldsymbol{x} < \infty$$

is called the *Orlicz class* and denoted by $\widetilde{L}_\Phi(\Omega)$.

Remark 1.16. Special cases of the Orlicz classes are Lebesgue spaces $L^p(\Omega)$, $p \geq 1$. We just put $\Phi(t) = ct^p$, where $c > 0$ is an arbitrary constant.

Theorem 1.17. *Let Φ and Ψ be a pair of complementary Young functions, $u \in \widetilde{L}_\Phi(\Omega)$ and $v \in \widetilde{L}_\Psi(\Omega)$. Then $uv \in L^1(\Omega)$ and*

$$\int_\Omega |u(\boldsymbol{x})v(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} \leq \int_\Omega \Phi(|u(\boldsymbol{x})|)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \Psi(|v(\boldsymbol{x})|)\,\mathrm{d}\boldsymbol{x}.$$

Proof: The assertion follows directly from Theorem 1.9.

Lemma 1.18. *Let Φ be a Young function. Then $\widetilde{L}_\Phi(\Omega)$ is a convex set and*

$$\widetilde{L}_\Phi(\Omega) \subset L^1(\Omega)$$

for $|\Omega| < \infty$.

Proof: See [2, page 130]

Definition 1.19. Let Φ, Ψ be a pair of complementary Young functions. The set of all Lebesgue-measurable functions u defined almost everywhere in Ω such that

$$\|u\|_{\Phi} := \sup\left\{ \int_{\Omega} |u(\boldsymbol{x})v(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}; v \in \widetilde{L}_{\Psi}(\Omega), \rho(v;\Psi) \leq 1 \right\} < \infty$$

is called the *Orlicz space* and denoted by $L_{\Phi}(\Omega)$.

Theorem 1.20. *Let Φ_1, Φ_2 be two Young functions. Then $L_{\Phi_1}(\Omega) \hookrightarrow L_{\Phi_2}(\Omega)$ if and only if $\Phi_2 \prec \Phi_1$.*

Proof: See [2, page 185].

Theorem 1.21 (Hölder's inequality)**.** *Let Φ, Ψ be a pair of complementary Young functions, $u \in L_{\Phi}(\Omega)$ and $v \in L_{\Psi}(\Omega)$. Then $uv \in L^1(\Omega)$ and*

$$\int_{\Omega} |u(\boldsymbol{x})v(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x} \leq \|u\|_{\Phi}\|v\|_{\Psi}.$$

Proof: See [2, str. 152].

Theorem 1.22. *Let Φ satisfy the Δ_2-condition. Then the Orlicz space $L_{\Phi}(\Omega)$ is separable.*

Proof: See [2, page 161].

Definition 1.23. The space $E_{\Phi}(\Omega)$ is defined to be the closure of $B(\Omega)$, i.e. of the set of all bounded measurable functions defined in Ω, with respect to the norm $\|\cdot\|_{\Phi}$.

Remark 1.24. It generally holds

$$E_{\Phi}(\Omega) \subseteq \widetilde{L}_{\Phi}(\Omega) \subseteq L_{\Phi}(\Omega),$$

where the equality occurs if and only if Φ satisfies the Δ_2-condition.

Theorem 1.25. *Let Φ_1, Φ_2 be two Young functions. If $\Phi_2 \prec\prec \Phi_1$, then $L_{\Phi_1}(\Omega) \hookrightarrow E_{\Phi_2}(\Omega)$.*

Proof: See [2, page 189].

Theorem 1.26. *$E_\Phi(\Omega)$ is a separable space.*

Proof: See [2, page 166].

Theorem 1.27. *Let F be a continuous linear functional in $E_\Phi(\Omega)$. Then there exists a uniquely determined function $v \in L_\Psi(\Omega)$ such that*

$$F(u) = \int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad \forall u \in E_\Phi(\Omega).$$

Proof: See [2, page 169].

Remark 1.28. The assertion of the foregoing theorem can be expressed as

$$L_\Psi(\Omega) = [E_\Phi(\Omega)]^*.$$

Definition 1.29. We say that a sequence $\{u_n\}_{n=1}^\infty \subset L_\Phi(\Omega)$ *converges E_Ψ-weakly* to $u \in L_\Phi(\Omega)$, denoted by $u_n \overset{\Psi}{\rightharpoonup} u$, for $n \to \infty$ if

$$\int_\Omega u_n v\,\mathrm{d}\boldsymbol{x} \to \int_\Omega uv\,\mathrm{d}\boldsymbol{x}$$

for any function $v \in E_\Psi(\Omega)$.

Remark 1.30. In view of Theorem 1.27 it is obvious that the E_Ψ-weak convergence (Definition 1.29) coincides with the weak-$*$ convergence (Definition 1.2). Furthermore, it follows from Theorem 1.3 and Theorem 1.26 that every Orlicz space $L_\Phi(\Omega)$ is E_Ψ-weakly compact.

Definition 1.31. Let Φ be a Young function, let X be a Banach space and Q be a nonempty bounded open subset of $\mathbb{R}^N$. Denote by $L_\Phi(Q;X)$ the set of all measurable mappings $u\colon Q \to X$ such that

$$\|u\|_{L_\Phi(Q;X)} := \sup\left\{\int_Q \|u(t)\|_X |v(t)|\,\mathrm{d}t; v \in \widetilde{L}_\Psi(Q), \rho(v;\Psi) \leq 1\right\} < \infty.$$

Definition 1.32. We say that a sequence $\{u_n\}_{n=1}^{\infty} \subset L_{\Phi_1}(0,T; L_{\Phi_2}(\Omega))$ *converges weak-∗* to $u \in L_{\Phi_1}(0,T; L_{\Phi_2}(\Omega))$ if

$$\int_0^T \varphi(t) \int_\Omega u_n(\boldsymbol{x},t)\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \to \int_0^T \varphi(t) \int_\Omega u(\boldsymbol{x},t)\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

for every $\psi \in E_{\Psi_2}(\Omega)$ and $\varphi \in E_{\Psi_1}(0,T)$.

Definition 1.33. We define *Sobolev-Orlicz space* $W^k L_\Phi(\Omega)$ as the space of all functions u such that

$$\|u\|_{k,\Phi} := \sqrt{\sum_{\alpha,|\alpha|\le k} \|\mathrm{D}^\alpha u\|_\Phi^2} < \infty,$$

the space $W^k E_\Phi(\Omega)$ as the closure of $C^\infty(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{k,\Phi}$ and the space $W_0^k L_\Phi(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the same norm. We denote

$$W^{-1}L_\Phi(\Omega) = \left[W_0^1 L_\Psi(\Omega)\right]^*,$$

where Ψ is the complementary function to Φ.

Definition 1.34. We say that a sequence $\{u_n\}_{n=1}^{\infty} \subset W^1 L_\Phi(\Omega)$ *converges E_Ψ-weakly* to $u \in W^1 L_\Phi(\Omega)$, $u_n \overset{\Psi}{\rightharpoonup} u$ in $W^1 L_\Phi(\Omega)$, if $u_n \overset{\Psi}{\rightharpoonup} u$ and simultaneously $\nabla u_n \overset{\Psi}{\rightharpoonup} \nabla u$.

Definition 1.35. Let us denote

$$K_\Phi := \left\{v \in L_\Phi(\mathbb{R}^N); v \ge 0, \partial_i v \in L_\Phi(\mathbb{R}^N), i = 1,\dots,N\right\}$$

and define the *Φ-capacity* of a set $M \subset \mathbb{R}^N$ as

$$\mathrm{cap}_\Phi(M) := \inf\left\{\int_{\mathbb{R}^N} \Phi(|\nabla v|)\,\mathrm{d}\boldsymbol{x}; v \in K_\Phi, v \ge 1 \text{ in } M\right\}.$$

Remark 1.36. Note that

$$\mathrm{cap}_\Phi(K) = \inf\left\{\int_{\mathbb{R}^N} \Phi(|\nabla v|)\,\mathrm{d}\boldsymbol{x}; v \in \mathcal{D}(\mathbb{R}^N), v \ge 1 \text{ in } M\right\}$$

for each compact $K \subset \mathbb{R}^N$. See [7, page 134].

Definition 1.37. Assume that Φ is a Young function. Let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of open sets in $\mathbb{R}^3$. We say that Ω_n *converges* to an open set $\Omega \subset \mathbb{R}^3$ with respect to Φ, denoted by $\Omega_n \xrightarrow{\Phi} \Omega$, if

- for any compact set $K \subset \Omega$ there exists $m \in \mathbb{N}$ such that

$$K \subset \Omega_n \quad \forall n \geq m, \tag{1.3}$$

- the sets $\Omega_n \setminus \Omega$ are bounded and

$$\mathrm{cap}_{\Phi}(\overline{\Omega_n \setminus \Omega}) \to 0 \text{ pro } n \to \infty. \tag{1.4}$$

1.3 Fast growing Young functions

Definition 1.38. We define the Young functions

$$\Phi_1(z) := z \ln(1+z),$$

$$\Phi_\gamma(z) := (1+z)\ln^\gamma(1+z) \quad \text{for } \gamma > 1.$$

and denote Ψ_1, Ψ_γ their complementary functions. Next we define the Young function

$$M(z) := \mathrm{e}^z - z - 1$$

and $\overline{M}$ stands for its complementary function. Let us denote $\Phi_{\frac{1}{\alpha}}(z)$ the Young functions with the asymptotic growth $z \ln^{\frac{1}{\alpha}}(z)$ for $z \geq z_0 > 0$, $\alpha \in (1, \infty)$, and $\Psi_{\frac{1}{\alpha}}(z)$ their complementary functions.

Definition 1.39. Let us define the spaces

$$X := \{\boldsymbol{v} \colon \Omega \to \mathbb{R}^N; \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}, \mathrm{D}\boldsymbol{v} \in L_M(\Omega)\},$$

$$Y := \{\boldsymbol{v} \colon \Omega \times (0,T) \to \mathbb{R}^N; \boldsymbol{v}(t)|_{\partial\Omega} = \boldsymbol{0} \text{ for a.a. } t \in (0,T), \mathrm{D}\boldsymbol{v} \in L_M(\Omega \times (0,T))\}$$

and their norms

$$\|\boldsymbol{v}\|_X := \|\mathrm{D}\boldsymbol{v}\|_{M,\Omega}, \quad \|\boldsymbol{v}\|_Y := \|\mathrm{D}\boldsymbol{v}\|_{M,\Omega\times(0,T)}.$$

Lemma 1.40. *Let Φ_γ be the Young functions established in Definition 1.38 for $\gamma > 0$. Then their complementary functions satisfy the estimates*

$$c_1 \mathrm{e}^{\left(\frac{z}{c}\right)^{\frac{1}{\gamma}}} \leq \Psi_\gamma(z) \leq c_2 \mathrm{e}^{2z^{\frac{1}{\gamma}}}$$

for $z \geq z_0(\gamma) > 0$, $c(\gamma) > 0$ and $c_i(\gamma) > 0$.

Proof: See [9, page 13].

Remark 1.41. It follows from the foregoing lemma, that Ψ_1 and M are equivalent. The same holds for Φ_1 and $\overline{M}$.

Lemma 1.42. *Young functions Φ_γ, $\gamma \geq 1$, satisfy the global Δ_2-condition.*

Proof: See [9, page 14].

Lemma 1.43. *Relationship between Young functions Φ_{γ_1} and Φ_{γ_2} is $\Phi_{\gamma_1} \prec\prec \Phi_{\gamma_2}$ for $0 < \gamma_1 < \gamma_2$. For their complementary functions it then holds $\Psi_{\gamma_2} \prec\prec \Psi_{\gamma_1}$.*

Proof: See [9, page 14].

Theorem 1.44 (Korn's inequality)**.** *Let $\boldsymbol{u} \in W_0^{1,p}(\Omega)$ for all $p > 1$. Then*

$$\|\boldsymbol{u}\|_{1,p} \leq \frac{cp^2}{p-1} \|\mathrm{D}\boldsymbol{u}\|_p.$$

Proof: See [4].

Lemma 1.45. *Let $u \in L_{\Psi_2}(\Omega)$, $v \in L_{\Psi_1}(\Omega)$ and*

$$\|u\|_p \leq cp\|v\|_p, \quad \forall p \geq 2,$$

where the constant c does not depend on p. Then

$$\|u\|_{\Psi_2} \leq c\|v\|_M.$$

Proof: See [9, page 17].

Theorem 1.46. *Let $u \in \widetilde{L}_M(\Omega\times(0,T))$. Then $u \in L_M(0,T;L_M(\Omega))$. Let further $v \in \widetilde{L}_{\Phi_1}(\Omega\times(0,T))$. Then $u \in L_{\Phi_{\frac{1}{\alpha}}}(0,T;L_{\Phi_{\frac{1}{\beta}}}(\Omega))$ for $\alpha,\beta \in (1,\infty)$ such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$.*

Proof: See [9, page 22].

Theorem 1.47. *Let $u \in \widetilde{L}_{\Psi_\gamma}(\Omega\times(0,T))$, $\gamma \geq 1$. Then $u \in L_{\Psi_\gamma}(0,T;L_{\Psi_\gamma}(\Omega))$.*

Proof: We proceed similarly as in [9, page 22]. For functions $\varphi \in L_{\Phi_\gamma}(\Omega)$, $\int_\Omega \Phi_\gamma(|\varphi|)\,\mathrm{d}\boldsymbol{x} \leq 1$, and $\psi \in L_{\Phi_\gamma}(0,T)$, $\int_0^T \Phi_\gamma(|\psi|)\,\mathrm{d}t \leq 1$ we have

$$\begin{aligned}
\sup_\psi \left|\int_0^T \psi \sup_\varphi \int_\Omega u\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| &\leq \int_0^T\!\!\int_\Omega \Psi_\gamma(u)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \\
&+ \sup_\psi \int_0^T \sup_\varphi \int_\Omega |\varphi\psi| \ln^\gamma(1+|\varphi\psi|)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + c \leq \\
&\leq c(u)\left(\sup_\psi \int_0^T \sup_\varphi \int_\Omega |\varphi\psi| \ln^\gamma\Big((1+|\varphi|)(1+|\psi|)\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + 1\right) \leq \\
&\leq c(u)\left(\sup_\psi \int_0^T \sup_\varphi \int_\Omega |\varphi\psi| \Big(\ln(1+|\varphi|)+\ln(1+|\psi|)\Big)^\gamma\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + 1\right) \leq \\
&\leq c(u)\left(\sup_\psi \int_0^T \sup_\varphi \int_\Omega |\varphi\psi| \ln^\gamma(1+|\varphi|)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \right.\\
&\left. + \sup_\psi \int_0^T \sup_\varphi \int_\Omega |\varphi\psi| \ln^\gamma(1+|\psi|)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + 1\right) = \\
&= c(u)\left(\sup_\psi \int_0^T |\psi|\,\mathrm{d}t \sup_\varphi \int_\Omega |\varphi| \ln^\gamma(1+|\varphi|)\,\mathrm{d}\boldsymbol{x} + \right.\\
&\left. + \sup_\psi \int_0^T |\psi|\ln^\gamma(1+|\psi|)\,\mathrm{d}t \sup_\varphi \int_\Omega |\varphi|\,\mathrm{d}\boldsymbol{x} + 1\right) < \infty,
\end{aligned}$$

where we have used the Young inequality and the Jensen inequality (1.1). □

2 Formulation of the problem and main results

We consider a system of equations describing flow of the compresible isothermal fluids with nonlinear stress tensor. This system is composed of the continuity equation

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0 \quad \text{in } \Omega \tag{2.1}$$

and the momentum equation

$$\partial_t(\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla \rho - \operatorname{div} \boldsymbol{S}(\mathrm{D}\boldsymbol{u}) = \rho \boldsymbol{f} \quad \text{in } \Omega, \tag{2.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$. The system is completed by the boundary condition

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{0}, \quad \boldsymbol{x} \in \Omega, t \in (0, T), T > 0, \tag{2.3}$$

and the initial conditions

$$\rho(\boldsymbol{x}, 0) = \rho_0(\boldsymbol{x}) \geq 0, \quad \boldsymbol{x} \in \Omega, \tag{2.4}$$

$$(\rho \boldsymbol{u})(\boldsymbol{x}, 0) = \boldsymbol{q}_0, \quad \boldsymbol{x} \in \Omega. \tag{2.5}$$

In addition, we assume that the stress tensor $\boldsymbol{S}$ satisfies these conditions:

1. $\boldsymbol{S}$ is coercive, i.e.

$$\int_\Omega \boldsymbol{S}(\mathrm{D}\boldsymbol{v}) : \mathrm{D}\boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \geq \int_\Omega M(|\mathrm{D}\boldsymbol{v}|) \, \mathrm{d}\boldsymbol{x} \tag{2.6}$$

   for any function $\boldsymbol{v} \in X$,

2. $\boldsymbol{S}$ is monotone, i.e.

$$\int_\Omega (\boldsymbol{S}(\mathrm{D}\boldsymbol{v}) - \boldsymbol{S}(\mathrm{D}\boldsymbol{w})) : (\mathrm{D}\boldsymbol{v} - \mathrm{D}\boldsymbol{w}) \varphi \, \mathrm{d}\boldsymbol{x} \geq 0 \tag{2.7}$$

   for any $\boldsymbol{v}, \boldsymbol{w} \in X$ and $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$,

3. $\boldsymbol{S}$ is bounded in the following sense:

$$\int_\Omega \overline{M}(\boldsymbol{S}(\mathrm{D}\boldsymbol{v})) \, \mathrm{d}\boldsymbol{x} \leq c \left(1 + \int_\Omega M(|\mathrm{D}\boldsymbol{v}|) \, \mathrm{d}\boldsymbol{x} \right) \tag{2.8}$$

   for any functions $\boldsymbol{v} \in X$ and let $\boldsymbol{S}(\boldsymbol{v} - \varepsilon \boldsymbol{w}) \xrightharpoonup{M} \boldsymbol{S}(\boldsymbol{v})$ for $\varepsilon \to 0$ and any function $\boldsymbol{v} \in Y$ such that $\mathrm{D}\boldsymbol{v} \in \widetilde{L}_M(\Omega \times (0, T))$ and any $\boldsymbol{w} \in C_0^\infty(\Omega \times (0, T))$,

4. $\mathcal{S}$ satisfies the estimate

$$\int_0^T\int_\Omega |\mathcal{S}(\mathrm{D}\boldsymbol{v}_1) - \mathcal{S}(\mathrm{D}\boldsymbol{v}_2)|\,\mathrm{d}\boldsymbol{x}\mathrm{d}t \le c(T,\kappa)\int_0^T \|\mathrm{D}\boldsymbol{v}_1(t) - \mathrm{D}\boldsymbol{v}_2(t)\|_\infty\,\mathrm{d}t$$

for $\boldsymbol{v}_i \in M_\kappa$, $i = 1, 2$, where

$$M_\kappa := \big\{\boldsymbol{v} \in C([0,T]; W_0^{1,2}(\Omega)) \cap L^\infty(0,T; W^{1,\infty}(\Omega)); \\ \|\boldsymbol{v}(t)\|_\infty + \|\nabla\boldsymbol{v}(t)\|_\infty \le \kappa \text{ for a.a. } t \in [0,T]\big\},$$

5. if $\{\mathrm{D}\boldsymbol{u}_n\}_{n=1}^\infty \subset \widetilde{L}_M(\Omega\times(0,T))$ is a sequence such that

$$\mathrm{D}\boldsymbol{u}_n \xrightarrow{\overline{M}} \mathrm{D}\boldsymbol{u} \quad \text{in } L_M(\Omega\times(0,T))$$

and

$$\int_0^T\int_\Omega \mathcal{S}(\mathrm{D}\boldsymbol{u}_n) : \mathrm{D}\boldsymbol{u}_n\,\mathrm{d}\boldsymbol{x}\mathrm{d}t \le c \quad \text{for all } n \in \mathbb{N},$$

then

$$\liminf_{n\to\infty}\int_0^t\int_\Omega \mathcal{S}(\mathrm{D}\boldsymbol{u}_n) : \mathrm{D}\boldsymbol{u}_n\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \ge \int_0^t\int_\Omega \mathcal{S}(\mathrm{D}\boldsymbol{u}) : \mathrm{D}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{2.9}$$

for all $t \in [0,T]$ and any Ω.

Note that such a tensor really exist. As an example we can take

$$\mathcal{S}(\mathrm{D}\boldsymbol{u}) = \begin{cases} \frac{M(|\mathrm{D}\boldsymbol{u}|)\mathrm{D}\boldsymbol{u}}{|\mathrm{D}\boldsymbol{u}|^2} & \text{for } \mathrm{D}\boldsymbol{u} \ne 0, \\ 0 & \text{for } \mathrm{D}\boldsymbol{u} = 0. \end{cases}$$

For the proof that it satisfies the above mentioned conditions see [9, page 43].

Definition 2.1. Couple $(\rho, \boldsymbol{u})$ is called the *variational solution* of system (2.1)-(2.5) if

- the density ρ is a nonnegative function,

- continuity equation (2.1) is satisfied in the in the sense of distribution in $\mathbb{R}^3$ and in the sense of renormalized solution, i.e.

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)\boldsymbol{u}) + (b'(\rho)\rho - b(\rho)) \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times (0,T)) \tag{2.10}$$

  for any $b \in C^1([0,\infty))$ such that b and b' are bounded provided ρ and $\boldsymbol{u}$ were extended to be zero outside Ω,

- $\boldsymbol{u}$ satisfies (2.3) in the sense of traces and equation (2.2) holds in space $\mathcal{D}'(\Omega \times (0,T))$,

- the energy inequality

$$E(\tau) + \int_0^\tau \int_\Omega \boldsymbol{S}(\boldsymbol{u}) : \mathrm{D}\boldsymbol{u} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le E(0) + \int_0^\tau \int_\Omega \rho \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{2.11}$$

  holds for a.a. $t \in [0,T]$, where

$$E(t) = \frac{1}{2} \int_\Omega \left(\rho(t)|\boldsymbol{u}(t)|^2 + \rho(t) \ln \rho(t)\right) \mathrm{d}\boldsymbol{x},$$

$$E(0) = \frac{1}{2} \int_\Omega \left(\frac{|\boldsymbol{q}_0|^2}{\rho_0} + \rho_0 \ln \rho_0\right) \mathrm{d}\boldsymbol{x},$$

- the initial conditions are satisfied in the sense

$$\lim_{t\to 0^+} \int_\Omega \rho(t)\eta \,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho_0 \eta \,\mathrm{d}\boldsymbol{x}, \quad \forall \eta \in \mathcal{D}(\Omega),$$

$$\lim_{t\to 0^+} \int_\Omega \rho(t)\boldsymbol{u}(t) \cdot \boldsymbol{\eta} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{q}_0 \cdot \boldsymbol{\eta} \,\mathrm{d}\boldsymbol{x}, \quad \forall \boldsymbol{\eta} \in \mathcal{D}(\Omega),$$

Theorem 2.2. *Assume that $\partial\Omega \in C^{2+\mu}$ and tensor $\boldsymbol{S}$ satisfies conditions 1.-5. Let $\boldsymbol{f} \in \widetilde{L}_{\Psi_{\frac{\beta}{2}}}(\Omega \times (0,T))$, $\beta > 2$. For given initial data $\rho_0 \in L_{\Phi_\beta}(\Omega)$ and $\boldsymbol{q}_0$ such that $\frac{|\boldsymbol{q}_0|^2}{\rho_0} := \boldsymbol{0}$ if $\rho_0 = 0$ and $\frac{|\boldsymbol{q}_0|^2}{\rho_0} \in L^1(\Omega)$ for $\rho_0 > 0$ there exist functions*

$$\rho \in L^\infty(0,T; L_{\Phi_\beta}(\Omega)), \quad \boldsymbol{u} \in Y$$

such that couple $(\rho, \boldsymbol{u})$ is the variational solution of (2.1)-(2.5). In addition

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{1}{2}\rho|\boldsymbol{u}|^2 + \rho\ln\rho\,\mathrm{d}\boldsymbol{x}\right) + \int_\Omega \boldsymbol{S}(\mathrm{D}\boldsymbol{u}) : \mathrm{D}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho\boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}$$

in $\mathcal{D}'(\Omega\times(0,T))$.

Proof: See [9, page 73].

The main result states:

Theorem 2.3. *Let $\{\Omega_n\}_{n=1}^\infty$ be a sequence of open sets in $\mathbb{R}^3$ such that $\Omega_n \xrightarrow{\Psi_2} \Omega$ (see Definition 1.37) where Ω is a nonempty open set. Assume that tensor $\boldsymbol{S}$ satisfies conditions 1.-5. Let $(\rho_n,\ \boldsymbol{u}_n)$ be a variational solution of the problem (2.1)-(2.5) in $\Omega_n\times(0,T)$ with the driving force $\boldsymbol{f}_n = \boldsymbol{f}\chi_{\Omega_n}$, $\boldsymbol{f}\in \widetilde{L}_{\Psi_{\frac{\beta}{2}}}(\Omega\times(0,T))$, $\beta > 2$, and initial data $\rho_0^n = \rho_0\chi_{\Omega_n}$, $\rho_0 \in L_{\Phi_\beta}(\Omega)$, and $\boldsymbol{q}_0^n = \boldsymbol{q}_0\chi_{\Omega_n}$ such that $\frac{|\boldsymbol{q}_0|^2}{\rho_0} := \boldsymbol{0}$ if $\rho_0 = 0$ and $\frac{|\boldsymbol{q}_0|^2}{\rho_0} \in L^1(\Omega)$ for $\rho_0 > 0$. Then (passing to subsequences if necessary) we have*

$$\rho_n \to \rho \quad in\ C(\langle 0,T\rangle; L^{\mathrm{weak}}_{\Phi_\beta}(\mathbb{R}^3)),$$

$$\boldsymbol{u}_n \stackrel{*}{\rightharpoonup} \boldsymbol{u}_n \quad in\ L_M(0,T; W^1_0 L_{\Psi_2}(\mathbb{R}^3)),$$

$$\mathrm{D}\boldsymbol{u}_n \stackrel{*}{\rightharpoonup} \mathrm{D}\boldsymbol{u} \quad in\ L_M(0,T; L_M(\mathbb{R}^3))$$

and

$$\rho_n\boldsymbol{u}_n \to \rho\boldsymbol{u} \quad in\ C(\langle 0,T\rangle; L^{\mathrm{weak}}_{\Phi_\beta}(\mathbb{R}^3)),$$

where $(\rho,\ \boldsymbol{u})$ is a variational solution of the problem (2.1)-(2.5) in $\Omega\times(0,T)$ driven by the force $\boldsymbol{f}$ with initial data ρ_0 and $\boldsymbol{q}_0$.

We assume that the stress tensor $\boldsymbol{S}$ satisfies all the conditions 1.-5. but in fact we do not use the fourth condition in the whole text. It is introduced only because of Theorem 2.2.

Corollary 2.4. *The conclusion of Theorem 2.2 then keeps valid even if $\partial\Omega \in C^{0,1}$.*

3 Auxiliary assertions

Lemma 3.1. *If* $\mathrm{D}\boldsymbol{u} \in \widetilde{L}_M(\Omega \times (0,T))$, *then* $\boldsymbol{u} \in L^q(0,T;L^\infty(\Omega))$ *for* $q \in [1,\infty)$.

Proof: First we show that $\mathrm{D}\boldsymbol{u} \in L^q(0,T;L^p(\Omega))$ for arbitrary $p \in [1,\infty)$. Indeed,

$$\int_\Omega |\mathrm{D}\boldsymbol{u}|^p \,\mathrm{d}\boldsymbol{x} \le c\left(\int_\Omega M^{\frac{1}{q}}(|\mathrm{D}\boldsymbol{u}|)\,\mathrm{d}\boldsymbol{x} + 1\right) \le c\left(\left(\int_\Omega M^{\frac{1}{q}}(|\mathrm{D}\boldsymbol{u}|)\,\mathrm{d}\boldsymbol{x}\right)^p + 1\right),$$

because $|z|^p \le \mathrm{e}^{\frac{|z|}{q}} + c$ and $|a| \le |a|^p + 1$, and thus we can write

$$\int_0^T \left(\int_\Omega |\mathrm{D}\boldsymbol{u}|^p \,\mathrm{d}\boldsymbol{x}\right)^{\frac{q}{p}} \mathrm{d}t \le c\left(\int_0^T \left(\int_\Omega M^{\frac{1}{q}}(|\mathrm{D}\boldsymbol{u}|)\,\mathrm{d}\boldsymbol{x}\right)^q \mathrm{d}t + 1\right) \le$$

$$\le c\left(|\Omega|^{q-1}\int_0^T\!\!\int_\Omega M(|\mathrm{D}\boldsymbol{u}|)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + 1\right),$$

where we have used the Jensen inequality (1.2). From the fact that $\mathrm{D}\boldsymbol{u} \in L^q(0,T;L^p(\Omega))$ it follows from Theorem 1.44 that $\boldsymbol{u} \in L^q(0,T;W_0^{1,p}(\Omega))$ for arbitrary $p \in [1,\infty)$. Since $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $p > N$, we get $\boldsymbol{u} \in L^q(0,T;L^\infty(\Omega))$. □

Lemma 3.2. *Let* $\mathrm{D}\boldsymbol{u} \in \widetilde{L}_M(\Omega \times (0,T))$ *and* $\rho \in L^\infty(0,T;L_{\Phi_\beta}(\Omega))$, $\beta > 2$, *be a solution of (2.1) in the sense of distributions. Then*

$$\int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho_0 \,\mathrm{d}\boldsymbol{x} \quad a.e. \ \in (0,T).$$

Proof: Let us recall the fact (see [9, page 45, Lemma 4.1]) that if we extend the functions ρ and $\boldsymbol{u}$ to be zero outside Ω, then the equation (2.1) is satisfied in the space $\mathcal{D}'(\mathbb{R}^3 \times (0,T))$. Similarly as in [1, page 5] we take a sequence

$$\varphi_j \in \mathcal{D}(\mathbb{R}^3), \quad \varphi_j \ge 0, \quad \sup_{\boldsymbol{x}\in\mathbb{R}^3} |\nabla\varphi_j(\boldsymbol{x})| < 1/j, \quad \varphi_j \uparrow 1 \text{ for } j \to \infty$$

and take the test functions of the form $\varphi_j(\boldsymbol{x})\psi(t)$ with $\psi \in \mathcal{D}(0,T)$ to deduce

$$\int_{\mathbb{R}^3} \rho(\tau)\varphi_j \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} \rho(0)\varphi_j \,\mathrm{d}\boldsymbol{x} + \int_0^\tau\!\!\int_{\mathbb{R}^3} \rho\boldsymbol{u}\cdot\nabla\varphi_j \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

for a.a. $\tau \in (0,T)$. Since $\rho \in L^\infty(0,T; L_{\Phi_\beta}(\mathbb{R}^3))$ and $\boldsymbol{u} \in L^2(0,T; L^\infty(\mathbb{R}^3))$, then $\rho\boldsymbol{u} \in L^2(0,T; L^1(\mathbb{R}^3))$. Indeed,

$$\int_0^T \left(\int_{\mathbb{R}^3} |\rho\boldsymbol{u}| \right)^2 \mathrm{d}t \le \int_0^T (\|\rho(t)\|_1 \|\boldsymbol{u}(t)\|_\infty)^2 \,\mathrm{d}t \le$$

$$\le \|\rho\|^2_{L^\infty(0,T;L^1(\mathbb{R}^3))} \int_0^T \|\boldsymbol{u}(t)\|^2_\infty \,\mathrm{d}t = \|\rho\|^2_{L^\infty(0,T;L^1(\mathbb{R}^3))} \|\boldsymbol{u}\|^2_{L^2(0,T;L^\infty(\mathbb{R}^3))}.$$

Recall that $\rho(t) \in L^1(\mathbb{R}^3)$ because it is extended to be zero outside Ω, which is a bounded domain. Thus for $j \to \infty$ we infer

$$\int_\Omega \rho(\tau)\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} \rho(\tau)\,\mathrm{d}\boldsymbol{x} = \lim_{j\to\infty} \left(\int_{\mathbb{R}^3} \rho(0)\varphi_j \,\mathrm{d}\boldsymbol{x} + \int_0^\tau \int_{\mathbb{R}^3} \rho\boldsymbol{u} \cdot \nabla\varphi_j \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right) =$$

$$= \lim_{j\to\infty} \int_{\mathbb{R}^3} \rho(0)\varphi_j \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} \rho(0)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho_0 \,\mathrm{d}\boldsymbol{x}.$$

□

Let us denote $B \subset \mathbb{R}^N$ an open ball such that $\Omega_n \subset B$ for every $n \in \mathbb{N}$.

Lemma 3.3. *Let* $\Omega_n \xrightarrow{\Psi_2} \Omega$ *and*

$$w_n \xrightarrow{\Phi_2} w \quad \textit{in } W_0^1 L_{\Psi_2}(B),$$

where $w_n \in W_0^1 L_{\Psi_2}(\Omega_n)$. *Then* $w \in W_0^1 L_{\Psi_2}(\Omega)$.

Proof: According to (1.4) there exist functions $\varphi_n \in \mathcal{D}(B)$ such that

$$0 \le \varphi_n \le 1, \quad \varphi \equiv 1 \text{ in } V_n(\overline{\Omega_n \setminus \Omega}), \quad \varphi_n \to 0 \text{ in } W_0^1 L_{\Psi_2}(B),$$

where $V_n(\overline{\Omega_n \setminus \Omega})$ is an open neighbourhood of $\overline{\Omega_n \setminus \Omega}$. Put

$$v_n = (1-\varphi_n)T_k(w_n),$$

where T_k are the cut-off functions,

$$T_k(z) = kT\left(\frac{z}{k}\right), \quad k \ge 1,$$

with $T \in C^\infty(\mathbb{R})$ such that $T(-z) = T(z)$ for every $z \in \mathbb{R}$, T being concave in $(0, \infty)$ and

$$T(z) = \begin{cases} z \text{ for } 0 \le z \le 1, \\ 2 \text{ for } z \ge 3. \end{cases}$$

Functions $T_k(w_n)$ are obviously bounded in $W_0^1 L_{\Psi_2}(B)$ thus (passing to subsequence if necessary)

$$T_k(w_n) \overset{\Phi_2}{\rightharpoonup} \overline{T_k(w)} \quad \text{in } W_0^1 L_{\Psi_2}(B).$$

Furthermore, it is known from [2, page 358] that $W_0^1 L_{\Psi_2}(B) \hookrightarrow\hookrightarrow L^p(B)$ for arbitrary $p \in \mathbb{N}$. Hence

$$T_k(w_n) \to \overline{T_k(w)} \quad \text{in } L^p(B).$$

Similarly

$$w_n \overset{\Phi_2}{\rightharpoonup} w \quad \text{in } W_0^1 L_{\Psi_2}(B)$$

and

$$w_n \to w \quad \text{v } L^p(B)$$

and thus it follows from the uniqueness of the limit that $\overline{T_k(w)} = T_k(w)$ and we do not have to pass to the subsequence. Now we show that $v_n \overset{\Phi_2}{\rightharpoonup} T_k(w)$ in $W_0^1 L_{\Psi_2}(B)$. For $\psi \in E_{\Phi_2}(B)$ one has

$$\left| \int_B ((1 - \varphi_n) T_k(w_n) - T_k(w)) \psi \, \mathrm{d}\boldsymbol{x} \right| \le$$
$$\le \left| \int_B \varphi_n T_k(w_n) \psi \, \mathrm{d}\boldsymbol{x} \right| + \left| \int_B (T_k(w_n) - T_k(w)) \psi \, \mathrm{d}\boldsymbol{x} \right|,$$

where

$$\left| \int_B \varphi_n T_k(w_n) \psi \, \mathrm{d}\boldsymbol{x} \right| \le 2k \|\varphi_n\|_{\Psi_2} \|\psi\|_{\Phi_2} \to 0 \quad \text{for } n \to \infty$$

and

$$\left| \int_B (T_k(w_n) - T_k(w)) \psi \, \mathrm{d}\boldsymbol{x} \right| \to 0 \quad \text{for } n \to \infty$$

by virtue of the E_{Φ_2}-weak convergence of $T_k(w_n)$ and

$$\left|\int_B \nabla((1-\varphi_n)T_k(w_n) - T_k(w))\psi \,\mathrm{d}\boldsymbol{x}\right| \leq$$

$$\leq \left|\int_B \nabla\varphi_n T_k(w_n)\psi \,\mathrm{d}\boldsymbol{x}\right| + \left|\int_B ((1-\varphi_n)\nabla T_k(w_n) - \nabla T_k(w))\psi \,\mathrm{d}\boldsymbol{x}\right| \leq$$

$$\leq \left|\int_B \nabla\varphi_n T_k(w_n)\psi \,\mathrm{d}\boldsymbol{x}\right| + \left|\int_B \varphi_n \nabla T_k(w_n)\psi \,\mathrm{d}\boldsymbol{x}\right| +$$

$$+ \left|\int_B (\nabla T_k(w_n) - \nabla T_k(w))\psi \,\mathrm{d}\boldsymbol{x}\right|,$$

where

$$\left|\int_B \nabla\varphi_n T_k(w_n)\psi \,\mathrm{d}\boldsymbol{x}\right| \leq 2k\|\nabla\varphi_n\|_{\Psi_2}\|\psi\|_{\Phi_2} \to 0 \quad \text{for } n\to\infty,$$

$$\left|\int_B \varphi_n \nabla T_k(w_n)\psi \,\mathrm{d}\boldsymbol{x}\right| \leq$$

$$\leq \left|\int_B \varphi_n(\nabla T_k(w_n) - \nabla T_k(w))\psi \,\mathrm{d}\boldsymbol{x}\right| + \left|\int_B \varphi_n T_k'(w)\nabla w\psi \,\mathrm{d}\boldsymbol{x}\right| \leq$$

$$\leq \left|\int_B \varphi_n(\nabla T_k(w_n) - \nabla T_k(w))\psi \,\mathrm{d}\boldsymbol{x}\right| + \|\varphi_n\|_\infty \|\nabla w\|_{\Psi_2}\|\psi\|_{\Phi_2} \to 0 \quad \text{for } n\to\infty$$

as a consequence of the E_{Φ_2}-weak convergence of $\nabla T_k(w_n)$ and the strong convergence of φ to zero in $L^\infty(\Omega)$, and

$$\left|\int_B (\nabla T_k(w_n) - \nabla T_k(w))\psi \,\mathrm{d}\boldsymbol{x}\right| \to 0 \quad \text{for } n\to\infty,$$

which follows from the E_{Φ_2}-weak convergence of $\nabla T_k(w_n)$. Consequently $T_k(w) \in W_0^1 L_{\Psi_2}(\Omega)$ because obviously $v_n \in W_0^1 L_{\Psi_2}(\Omega)$ for every $n \in \mathbb{N}$. But this also means that $w \in W_0^1 L_{\Psi_2}(\Omega)$. □

Remark 3.4. If $\Omega_n \xrightarrow{\Psi_2} \Omega$ and

$$\mathrm{D}w_n \xrightarrow{\overline{M}} \mathrm{D}w \quad \text{in } L_M(B),$$

where $\mathrm{D}w_n \in L_M(\Omega_n)$, then it follows from Lemma 1.45 that (passing to subsequence if necessary)

$$w_n \xrightharpoonup{\Phi_2} w \quad \text{in } W_0^1 L_{\Psi_2}(B)$$

and according to the previous lemma $w \in W_0^1 L_{\Psi_2}(\Omega)$. Thus $\mathrm{D}w \in L_M(\Omega)$.

4 Apriori estimates

Let us denote

$$\theta_k(z) := T_k(\Phi_\beta(z)),$$

where $T_k(z)$ are the cut-off functions defined on page 22. Consider the equation (2.1) and put $b(\rho) = \theta_k'(\rho)$ in the renormalized continuity equation to infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta_k(\rho(t))\,\mathrm{d}\boldsymbol{x} - \int_\Omega \Big(\theta_k(\rho(t)) - \theta_k'(\rho(t))\Big)\operatorname{div}\boldsymbol{u}(t)\,\mathrm{d}\boldsymbol{x} = 0.$$

Now letting $k \to \infty$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi_\beta(\rho(t))\,\mathrm{d}\boldsymbol{x} - \int_\Omega \Big(\Phi_\beta(\rho(t)) - \rho(t)\Phi_\beta'(\rho(t))\Big)\operatorname{div}\boldsymbol{u}(t)\,\mathrm{d}\boldsymbol{x} = 0.$$

In the next step we use the fact that $\varepsilon M\left(\frac{z}{c}\right)$ and $\varepsilon\overline{M}(2cz)$ are for arbitrary ε and c complementary Young functions, $\overline{M}$ is equivalent with Φ_1 (see Lemma 1.41) and Φ_1 satisfies the Δ_2-condition (see Lemma 1.42). Employing to inequality

$$\Phi_1\Big(z\Phi_\gamma'(z) - \Phi_\gamma(z)\Big) \le \Phi_\gamma(z) + C,$$

see [9, page 56], we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi_\beta(\rho(t))\,\mathrm{d}\boldsymbol{x} = \int_\Omega \Big(\Phi_\beta(\rho(t)) - \rho(t)\Phi_\beta'(\rho(t))\Big)\operatorname{div}\boldsymbol{u}(t)\,\mathrm{d}\boldsymbol{x} \le \\
&\quad\le c(\varepsilon)\int_\Omega \Phi_1\Big(\Phi_\beta(\rho(t)) - \rho(t)\Phi_\beta'(\rho(t))\Big)\,\mathrm{d}\boldsymbol{x} + \varepsilon\int_\Omega M\left(\frac{|\operatorname{div}\boldsymbol{u}(t)|}{c}\right)\mathrm{d}\boldsymbol{x} \le \\
&\quad\le c(\varepsilon)\left(\int_\Omega \Phi_\beta(\rho(t))\,\mathrm{d}\boldsymbol{x} + 1\right) + \varepsilon\int_\Omega M\left(\frac{|\operatorname{div}\boldsymbol{u}(t)|}{c}\right)\mathrm{d}\boldsymbol{x} \\
&\quad\le c(\varepsilon)\left(\int_\Omega \Phi_\beta(\rho(t))\,\mathrm{d}\boldsymbol{x} + 1\right) + \varepsilon\int_\Omega M\left(|\mathrm{D}\boldsymbol{u}(t)|\right)\mathrm{d}\boldsymbol{x},
\end{aligned}$$

where the constant c is taken from the inequality

$$|\operatorname{div}\boldsymbol{u}| \leq c|\mathrm{D}\boldsymbol{u}|.$$

We add the obtained inequality to (2.11) and estimate the remaining term on the right-hand side as follows

$$\begin{aligned}\int_0^\tau\!\!\int_\Omega \rho\boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &= \int_0^\tau\!\!\int_\Omega \sqrt{\rho}\boldsymbol{f}\cdot\sqrt{\rho}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq \\ &\leq \int_0^\tau\!\!\int_\Omega \rho|\boldsymbol{f}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau\!\!\int_\Omega \rho|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq \\ &\leq \int_0^\tau\!\!\int_\Omega \varPhi_\beta(\rho)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau\!\!\int_\Omega \varPsi_\beta(|\boldsymbol{f}|^2)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau\!\!\int_\Omega \rho|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq \\ &\leq C + \int_0^\tau\!\!\int_\Omega \varPhi_\beta(\rho)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau\!\!\int_\Omega \rho|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned}$$

Using (2.6) we arrive to

$$\begin{aligned}\frac{1}{2}\int_\Omega \big(\rho(\tau)|\boldsymbol{u}(\tau)|^2 + \rho(\tau)\ln\rho(\tau)\big)\,\mathrm{d}\boldsymbol{x} + (1-\varepsilon)\int_0^\tau\!\!\int_\Omega M(|\mathrm{D}\boldsymbol{u}|)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \\ + \int_\Omega \varPhi_\beta(\rho(\tau))\,\mathrm{d}\boldsymbol{x} \leq \int_0^\tau\!\!\int_\Omega \rho|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + c(\varepsilon)\int_0^\tau\!\!\int_\Omega \varPhi_\beta(\rho(t))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + C(\boldsymbol{f}).\end{aligned}$$

From the integral Gronwall inequality we have

$$\frac{1}{2}\int_\Omega \rho(\tau)|\boldsymbol{u}(\tau)|^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega \varPhi_\beta(\rho(\tau))\,\mathrm{d}\boldsymbol{x} \leq C(\varepsilon,T).$$

Altogether

$$\begin{aligned}\frac{1}{2}\int_\Omega \big(\rho(\tau)|\boldsymbol{u}(\tau)|^2 + \rho(\tau)\ln\rho(\tau)\big)\,\mathrm{d}\boldsymbol{x} + (1-\varepsilon)\int_0^\tau\!\!\int_\Omega M(|\mathrm{D}\boldsymbol{u}|)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \\ + \int_\Omega \varPhi_\beta(\rho(\tau))\,\mathrm{d}\boldsymbol{x} \leq C(\varepsilon,T,\boldsymbol{f}).\end{aligned}$$

Consequently

$$\int_\Omega \rho(t)|\boldsymbol{u}(t)|^2\,\mathrm{d}\boldsymbol{x} \leq C(\varepsilon,T,\boldsymbol{f}), \quad \text{for a.a. } t\in[0,T], \tag{4.1}$$

$$\int_\Omega \rho(t)\ln\rho(t)\,\mathrm{d}\boldsymbol{x} \le C(\varepsilon, T, \boldsymbol{f}), \quad \text{for a.a. } t \in [0,T], \tag{4.2}$$

$$\int_\Omega \Phi_\beta(\rho(t))\,\mathrm{d}\boldsymbol{x} \le C(\varepsilon, T, \boldsymbol{f}), \quad \text{for a.a. } t \in [0,T], \tag{4.3}$$

$$\int_0^T\int_\Omega M(\mathrm{D}\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le C(\varepsilon, T, \boldsymbol{f}). \tag{4.4}$$

The estimates remain valid even if we replace ρ, $\boldsymbol{u}$, $\boldsymbol{f}$ and Ω with ρ_n, $\boldsymbol{u}_n$, $\boldsymbol{f}_n$ and Ω_n.

5 Limit passages

5.1 Continuity equation

Consider a sequence $\{(\rho_n, \boldsymbol{u}_n)\}_{n=1}^\infty$ of variational solutions of the problem (2.1)-(2.5) on corresponding sets Ω_n. We extend this functions to be zero in $\mathbb{R}^3 \setminus \Omega_n$ and take arbitrary open ball $B \subset \mathbb{R}^3$ such that $\Omega_n \subset B$ for all $n \ge m$. We can see from (4.3) that (passing to subsequence if necessary)

$$\rho_n \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(0,T; L_{\Phi_\beta}(B)). \tag{5.1}$$

Similarly we get from (4.4) and Theorem 1.46 that (passing to a subsequence if necessary)

$$\mathrm{D}\boldsymbol{u}_n \overset{*}{\rightharpoonup} \mathrm{D}\boldsymbol{u} \quad \text{in } L_M(0,T; L_M(B)). \tag{5.2}$$

Moreover, we know from Lemma 1.45 that

$$\boldsymbol{u}_n \overset{*}{\rightharpoonup} \boldsymbol{u} \quad \text{in } L_M(0,T; W_0^1 L_{\Psi_2}(B)), \tag{5.3}$$

where $\boldsymbol{u} \in L_M(0,T; W_0^1 L_{\Psi_2}(\Omega))$ from Lemma 3.3.

Recall that $\boldsymbol{u}_n$ are extended to be zero outside Ω_n, thus

$$\|\boldsymbol{u}_n\|_{L^2(0,T;L^\infty(B))} \le \|\boldsymbol{u}_n\|_{L^2(0,T;L^\infty(\Omega_n))} \le c\|\mathrm{D}\boldsymbol{u}_n\|_{L_M(0,T;L_M(\Omega_n))} < C(\varepsilon, T, \boldsymbol{f}),$$

which follows from (4.4) and the proof of Theorem 1.47. From continuity equation (2.1) we obtain for $\varphi \in W_0^1 L_{\Psi_\beta}(B)$ and $\psi \in L^2(0,T)$ the estimate

$$\left|\int_0^T \psi(t)\langle \partial_t \rho_n, \varphi\rangle \,\mathrm{d}t\right| = \left|\int_0^T \psi(t)\int_B \rho_n \boldsymbol{u}_n \cdot \nabla\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \leq$$
$$\leq \int_0^T |\psi(t)| \|\rho_n(t)\|_{\Phi_\beta} \|\boldsymbol{u}_n(t)\|_\infty \|\nabla\varphi\|_{\Psi_\beta}\,\mathrm{d}t \leq$$
$$\leq \|\psi\|_2 \|\rho_n\|_{L^\infty(0,T;L_{\Phi_\beta}(B))} \|\boldsymbol{u}_n\|_{L^2(0,T;L^\infty(B))} \|\nabla\varphi\|_{\Psi_\beta},$$

where we have used the conclusion of Lemma 3.1. Hence $\partial_t \rho_n$ are uniformly bounded in $L^2(0,T;W^{-1}L_{\Phi_\beta}(B))$. Since ρ_n are uniformly bouded in the space $L^\infty(0,T;L_{\Phi_\beta}(B))$ and

$$W_0^1 L_{\Psi_\beta}(B) \hookrightarrow W_0^{1,p}(B) \hookrightarrow\hookrightarrow C(\overline{B}) \hookrightarrow E_{\Psi_\beta}(B), \quad p > N,$$

i.e.

$$L_{\Phi_\beta}(B) \hookrightarrow\hookrightarrow W^{-1}L_{\Phi_\beta}(B),$$

one has from [8, page 85] that (passing to a subsequence if necessary)

$$\rho_n \to \rho \quad \text{in } C(\langle 0,T\rangle; W^{-1}L_{\Phi_\beta}(B))$$

and

$$\rho_n \to \rho \quad \text{in } C(\langle 0,T\rangle; L_{\Phi_\beta}^{\text{weak}}(B)), \tag{5.4}$$

i.e.

$$\left\|\int_B (\rho_n - \rho)\psi\,\mathrm{d}\boldsymbol{x}\right\|_{C([0,T])} \to 0 \quad forn \to \infty$$

for every $\psi \in E_{\Psi_\beta}(B)$. Since ρ_n are extended to be zero outside Ω_n, the same holds in $C(\langle 0,T\rangle; L_{\Phi_\beta}^{\text{weak}}(\mathbb{R}^3))$.

Now we are going to show the weak-$*$ convergence of functions $\rho_n \boldsymbol{u}_n$ to $\rho\boldsymbol{u}$.

At first we deduce for $\boldsymbol{\varphi} \in \widetilde{L}_{\Psi_{\frac{\beta}{2}}}(B)$ the estimate

$$\left|\int_B \rho_n(t)\boldsymbol{u}_n(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\right| = \left|\int_B \sqrt{\rho_n(t)}\boldsymbol{u}_n(t)\cdot\sqrt{\rho_n(t)}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\right| \leq$$

$$\leq \int_B \rho_n(t)|\boldsymbol{u}_n(t)|^2\,\mathrm{d}\boldsymbol{x} + \int_B \rho_n(t)|\boldsymbol{\varphi}|^2\,\mathrm{d}\boldsymbol{x} \leq$$

$$\leq c + \int_B \Phi_\beta(\rho_n(t))\,\mathrm{d}\boldsymbol{x} + \int_B \Psi_\beta(|\boldsymbol{\varphi}|^2)\,\mathrm{d}\boldsymbol{x},$$

for a.a. $t \in [0,T]$, where we have used estimate (4.1). Therefore

$$\rho_n\boldsymbol{u}_n \overset{*}{\rightharpoonup} \overline{\rho\boldsymbol{u}} \quad \text{in } L^\infty(0,T;L_{\Phi_{\frac{\beta}{2}}}(B)).$$

It remains to show that $\overline{\rho\boldsymbol{u}} = \rho\boldsymbol{u}$. Take arbitrary open ball $B_1 \subset \overline{B_1} \subset \Omega$ (this implies $\overline{B_1} \subset \Omega_n$ for every $n \geq m$). For $\boldsymbol{\varphi} \in W_0^1 L_{\Psi_\beta}(B_1)$ and $\psi \in L_{\Phi_1}(0,T)$ it holds

$$\left|\int_0^T \psi(t)\int_{B_1}(\rho_n\boldsymbol{u}_n - \rho\boldsymbol{u})\cdot\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \leq$$

$$\leq \left|\int_0^T \psi(t)\int_{B_1}(\rho_n - \rho)\boldsymbol{u}_n\cdot\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| + \left|\int_0^T \psi(t)\int_{B_1}\rho(\boldsymbol{u}_n - \boldsymbol{u})\cdot\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right|.$$

For the first integral we have

$$\left|\int_0^T \psi(t)\int_{B_1}(\rho_n - \rho)\boldsymbol{u}_n\cdot\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \leq$$

$$\leq \int_0^T |\psi(t)|\|\rho_n(t) - \rho(t)\|_{W^{-1}L_{\Phi_\beta}(B_1)}\|\boldsymbol{u}_n(t)\cdot\boldsymbol{\varphi}\|_{W_0^1 L_{\Psi_\beta}(B_1)}\,\mathrm{d}t \leq (*),$$

where

$$\|\boldsymbol{u}_n(t)\cdot\boldsymbol{\varphi}\|_{W_0^1 L_{\Psi_\beta}(B_1)} = \|\boldsymbol{u}_n(t)\cdot\boldsymbol{\varphi}\|_{\Psi_\beta,B_1} + \|\nabla(\boldsymbol{u}_n(t)\cdot\boldsymbol{\varphi})\|_{\Psi_\beta,B_1} \leq$$

$$\leq \|\boldsymbol{u}_n(t)\|_{\infty,\Omega_n}\|\boldsymbol{\varphi}\|_{\Psi_\beta,B_1} + \|\nabla\boldsymbol{u}_n(t)\boldsymbol{\varphi}\|_{\Psi_\beta,B_1} + \|\boldsymbol{u}_n(t)\nabla\boldsymbol{\varphi}\|_{\Psi_\beta,B_1} \leq$$

$$\leq \|\mathrm{D}\boldsymbol{u}_n(t)\|_{M,\Omega_n}\|\boldsymbol{\varphi}\|_{\Psi_\beta,B_1} + \|\nabla\boldsymbol{u}_n(t)\|_{\Psi_\beta,\Omega_n}\|\boldsymbol{\varphi}\|_{\infty,B_1} + \|\boldsymbol{u}_n(t)\|_{\infty,\Omega_n}\|\nabla\boldsymbol{\varphi}\|_{\Psi_\beta,B_1} \leq$$

$$\leq \|\mathrm{D}\boldsymbol{u}_n(t)\|_{M,\Omega_n}(\|\boldsymbol{\varphi}\|_{\Psi_\beta,B_1} + \|\boldsymbol{\varphi}\|_{\infty,B_1} + \|\nabla\boldsymbol{\varphi}\|_{\Psi_\beta,B_1}) \leq$$

$$\leq c\|\mathrm{D}\boldsymbol{u}_n(t)\|_{M,\Omega_n}\|\boldsymbol{\varphi}\|_{W_0^1 L_{\Psi_\beta}(B_1)},$$

where we have used the assumption that $\beta > 2$ and Lemma 1.45, thus

$$(*) \leq c\|\psi(t)\|_{\Phi_1}\|\rho_n - \rho\|_{C(\langle 0,T\rangle; W^{-1}L_{\Phi_\beta}(B_1))}\|\mathrm{D}\boldsymbol{u}_n\|_{L_M(0,T;L_M(\Omega_n))}\|\boldsymbol{\varphi}\|_{W_0^1 L_{\Psi_\beta}(B_1)}.$$

In the case of the second integral we use weak-$*$ convergence (5.3). It only remains to check that $\rho(t)\boldsymbol{\varphi} \in E_{\Phi_2}(B_1) = L_{\Phi_2}(B_1)$ (Φ_2 satisfies the Δ_2-condition). But this is easy because for $\sigma \in L_{\Psi_2}(B_1)$ it holds

$$\int_B \rho(t)\boldsymbol{\varphi}\sigma \,\mathrm{d}\boldsymbol{x} \leq c\|\rho\|_{L^\infty(0,T;L_{\Phi_\beta}(B_1))}\|\boldsymbol{\varphi}\|_\infty\|\sigma\|_{\Psi_2}.$$

Altogether

$$\rho_n\boldsymbol{u}_n \overset{*}{\rightharpoonup} \rho\boldsymbol{u} \quad \text{in } L_M(0,T;W^{-1}L_{\Phi_\beta}(B_1))$$

for arbitrary $B_1 \subset \overline{B_1} \subset \Omega$ and consequently from the uniquenes of the limit, the definition of open sets and the prolongation of ρ_n, $\boldsymbol{u}_n$, ρ and $\boldsymbol{u}$ function

$$\rho_n\boldsymbol{u}_n \overset{*}{\rightharpoonup} \rho\boldsymbol{u} \quad \text{in } L^\infty(0,T;L_{\Phi_{\frac{\beta}{2}}}(B)).$$

Since the ball B was arbitrary, we have deduced that couple $(\rho,\boldsymbol{u})$ satisfies the equation

$$\partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times (0,T)) \tag{5.5}$$

and also in the sense of renormalized solution (see [9, Lemma 4.2, page 46]).

5.2 Momentum equation

In the foregoing part we assumed that the open ball $B \subset \mathbb{R}^3$ satisfies $\Omega_n \subset B$ for $n \geq m$. But here we will consider the case $B \subset \overline{B} \subset \Omega_n$.

Now we are going to show the uniform boundedness of $\rho_n\boldsymbol{u}_n \otimes \boldsymbol{u}_n$ in the space $L^q(0,T;L_{\Phi_\beta}(B))$, $q \in [1,\infty)$. Let $\varphi \in \widetilde{L}_{\Psi_\beta}(B)$ and $\psi \in L^{q'}(0,T)$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then according to Lemma 3.1 and (4.4) for $\boldsymbol{u} = \boldsymbol{u}_n$ and $\Omega = \Omega_n$ it follows

$$\begin{aligned}\left|\int_0^T \psi(t)\int_B \rho_n|\boldsymbol{u}_n|^2\varphi(x)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| &\leq \int_0^T |\psi(t)|\|\rho_n(t)\|_{\Phi_\beta}\|\boldsymbol{u}_n(t)\|_\infty^2\|\varphi\|_{\Psi_\beta}\,\mathrm{d}t \leq \\ &\leq \|\psi\|_{q'}\|\rho_n\|_{L^\infty(0,T;L_{\Phi_\beta}(B))}\|\boldsymbol{u}_n\|^2_{L^{2q}(0,T;L^\infty(B))}\|\varphi\|_{\Psi_\beta} \leq \\ &\leq \|\psi\|_{q'}\|\rho_n\|_{L^\infty(0,T;L_{\Phi_\beta}(B))}\|\boldsymbol{u}_n\|^2_{L^{2q}(0,T;L^\infty(\Omega_n))}\|\varphi\|_{\Psi_\beta}.\end{aligned} \tag{5.6}$$

In the next step we obtain from the momentum equation for $\boldsymbol{\varphi} \in W_0^1 L_{\Psi_{\frac{1}{2}}}(B)$ and $\psi \in \widetilde{L}_{\Psi_{\frac{1}{2}}}(0,T)$ the estimate

$$\left|\int_0^T \psi(t)\langle \partial_t(\rho_n \boldsymbol{u}_n), \boldsymbol{\varphi}(x)\rangle \,\mathrm{d}t\right| \leq \underbrace{\left|\int_0^T \psi(t)\int_B (\rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n) : \mathrm{D}\boldsymbol{\varphi}(x)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right|}_{I_1} +$$

$$+ \underbrace{\left|\int_0^T \psi(t)\int_B \rho_n \operatorname{div}\boldsymbol{\varphi}(x)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right|}_{I_2} + \underbrace{\left|\int_0^T \psi(t)\int_B \boldsymbol{S}(\mathrm{D}\boldsymbol{u}_n) : \mathrm{D}\boldsymbol{\varphi}(x)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right|}_{I_3} +$$

$$+ \underbrace{\left|\int_0^T \psi(t)\int_B \rho_n \boldsymbol{f}_n \cdot \boldsymbol{\varphi}(x)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right|}_{I_4}.$$

We now estimate integrals I_i, $i = 1,2,3,4$, one by one:

$$\begin{aligned} I_1 &= \left|\int_0^T \psi(t)\int_B (\rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n) : \mathrm{D}\boldsymbol{\varphi}(x)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \leq \\ &\leq \int_0^T |\psi(t)| \|\rho_n(t)\boldsymbol{u}_n(t) \otimes \boldsymbol{u}_n(t)\|_{\Phi_\beta} \|\mathrm{D}\boldsymbol{\varphi}\|_{\Psi_\beta}\,\mathrm{d}t \leq \\ &\leq \|\psi\|_{q'} \|\rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n\|_{L^q(0,T;L_{\Phi_\beta}(B))} \|\mathrm{D}\boldsymbol{\varphi}\|_{\Psi_\beta}, \end{aligned}$$

because we have already proved that $\rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n$ are uniformly bounded in the space $L^q(0,T;L_{\Phi_\beta}(B))$,

$$\begin{aligned} I_2 &= \left|\int_0^T \psi(t)\int_B \rho_n \operatorname{div}\boldsymbol{\varphi}(x)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \leq \int_0^T |\psi(t)| \|\rho_n(t)\|_{\Phi_\beta} \|\operatorname{div}\boldsymbol{\varphi}\|_{\Psi_\beta}\,\mathrm{d}t \leq \\ &\leq \|\psi\|_1 \|\rho_n\|_{L^\infty(0,T;L_{\Phi_\beta}(B))} \|\operatorname{div}\boldsymbol{\varphi}\|_{\Psi_\beta}, \end{aligned}$$

where we have used the fact that $\beta > 2$, and therefore $\widetilde{L}_{\Psi_{\frac{1}{2}}}(B) \subset \widetilde{L}_{\Psi_\beta}(B)$,

$$\begin{aligned} I_3 &= \left|\int_0^T \psi(t)\int_B \boldsymbol{S}(\mathrm{D}\boldsymbol{u}_n) : \mathrm{D}\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \leq \\ &\leq \int_0^T |\psi(t)| \|\boldsymbol{S}(\mathrm{D}\boldsymbol{u}_n(t))\|_{\Phi_{\frac{1}{2}}} \|\mathrm{D}\boldsymbol{\varphi}\|_{\Psi_{\frac{1}{2}}}\,\mathrm{d}t \leq \end{aligned}$$

$$\leq \|\psi\|_{\Psi_{\frac{1}{2}}} \|\mathbf{S}(\mathrm{D}\boldsymbol{u}_n)\|_{L_{\Phi_{\frac{1}{2}}}(0,T;L_{\Phi_{\frac{1}{2}}}(B))} \|\mathrm{D}\boldsymbol{\varphi}\|_{\Psi_{\frac{1}{2}}} \leq$$

$$\leq \|\psi\|_{\Psi_{\frac{1}{2}}} \|\mathrm{D}\boldsymbol{\varphi}\|_{\Psi_{\frac{1}{2}}} c_1 \left(\int_0^T\!\!\int_B \overline{M}(|\mathbf{S}(\mathrm{D}\boldsymbol{u}_n)|)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + 1 \right) \leq$$

$$\leq \|\psi\|_{\Psi_{\frac{1}{2}}} \|\mathrm{D}\boldsymbol{\varphi}\|_{\Psi_{\frac{1}{2}}} c_2 \left(\int_0^T\!\!\int_B M(|\mathrm{D}\boldsymbol{u}_n|)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + 1 \right)$$

$$\leq \|\psi\|_{\Psi_{\frac{1}{2}}} \|\mathrm{D}\boldsymbol{\varphi}\|_{\Psi_{\frac{1}{2}}} c_2 \left(\int_0^T\!\!\int_{\Omega_n} M(|\mathrm{D}\boldsymbol{u}_n|)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + 1 \right)$$

from Theorem 1.46 and the third assumption on the stress tensor $\mathbf{S}$,

$$I_4 = \left| \int_0^T \psi(t) \int_B \rho_n \boldsymbol{f}_n \cdot \boldsymbol{\varphi}(x)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right| \leq \int_0^T |\psi(t)| \|\rho_n(t)\|_{\Phi_\beta} \|\boldsymbol{f}_n(t)\|_{\Psi_\beta} \|\boldsymbol{\varphi}\|_\infty \,\mathrm{d}t \leq$$
$$\leq c \|\psi\|_{\Phi_{\frac{\beta}{2}}} \|\rho_n\|_{L^\infty(0,T;L_{\Phi_\beta}(B))} \|\boldsymbol{f}_n\|_{L_{\Psi_{\frac{\beta}{2}}}(0,T;L_{\Psi_{\frac{\beta}{2}}}(B))} \|\boldsymbol{\varphi}\|_\infty$$

in view of Theorem 1.47 and the fact that $L_{\Psi_{\frac{\beta}{2}}}(B) \hookrightarrow L_{\Psi_\beta}$, which dives the result that $\partial_t(\rho_n \boldsymbol{u}_n)$ are uniformly bouded in $L_{\Phi_{\frac{1}{2}}}(0,T;W^{-1}L_{\Phi_{\frac{1}{2}}}(B))$. Since $\rho_n \boldsymbol{u}_n$ are moreover uniformly bounded in $L^\infty(0,T;L_{\Phi_{\frac{\beta}{2}}}(B))$ and

$$W_0^1 L_{\Psi_{\frac{1}{2}}}(B) \hookrightarrow W_0^1 L_{\Psi_\beta}(B) \hookrightarrow W_0^{1,p}(B) \hookrightarrow\hookrightarrow C(\overline{B}) \hookrightarrow E_{\Psi_{\frac{1}{2}}}(B),$$

i.e.

$$L_{\Phi_{\frac{1}{2}}}(B) \hookrightarrow\hookrightarrow W^{-1} L_{\Phi_\beta}(B) \hookrightarrow W^{-1} L_{\Phi_{\frac{1}{2}}}(B)),$$

we can write (see [8, page 85])

$$\rho_n \boldsymbol{u}_n \to \rho \boldsymbol{u} \quad \text{in } C(\langle 0,T\rangle; W^{-1} L_{\Phi_\beta}(B)). \tag{5.7}$$

Now we are going to show the convergence of $\rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n$. For $\boldsymbol{\varphi} \in W_0^1 L_{\Psi_\beta}(B)$ and $\psi \in L_{\Phi_1}(0,T)$ we have

$$\left| \int_0^T \psi(t) \int_B (\rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n - \rho \boldsymbol{u} \otimes \boldsymbol{u}) : \boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right| \leq$$
$$\leq \left| \int_0^T \psi(t) \int_B ((\rho_n \boldsymbol{u}_n - \rho \boldsymbol{u}) \otimes \boldsymbol{u}_n) : \boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right| +$$
$$+ \left| \int_0^T \psi(t) \int_B (\rho \boldsymbol{u} \otimes (\boldsymbol{u}_n - \boldsymbol{u})) : \boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right|.$$

For the first integral

$$\left|\int_0^T \psi(t)\int_B ((\rho_n \boldsymbol{u}_n - \rho\boldsymbol{u})\otimes \boldsymbol{u}_n) : \boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \leq$$

$$\leq \int_0^T |\psi(t)| \|\rho_n(t)\boldsymbol{u}_n(t) - \rho(t)\boldsymbol{u}(t)\|_{W^{-1}L_{\Phi_\beta}(B)} \|\boldsymbol{u}_n(t)\boldsymbol{\varphi}\|_{W_0^1 L_{\Psi_\beta}(B)}\,\mathrm{d}t \leq (*),$$

whereas

$$\begin{aligned}
&\|\boldsymbol{u}_n(t)\boldsymbol{\varphi}\|_{W_0^1 L_{\Psi_\beta}(B)} = \|\boldsymbol{u}_n(t)\boldsymbol{\varphi}\|_{\Psi_\beta,B} + \|\nabla(\boldsymbol{u}_n(t)\boldsymbol{\varphi})\|_{\Psi_\beta,B} \leq\\
&\leq \|\boldsymbol{u}_n(t)\|_{\infty,\Omega_n}\|\boldsymbol{\varphi}\|_{\Psi_\beta,B} + \|\nabla\boldsymbol{u}_n(t)\boldsymbol{\varphi}\|_{\Psi_\beta,B} + \|\boldsymbol{u}_n(t)\operatorname{div}\boldsymbol{\varphi}\|_{\Psi_\beta,B} \leq\\
&\leq \|\mathrm{D}\boldsymbol{u}_n(t)\|_{M,\Omega_n}\|\boldsymbol{\varphi}\|_{\Psi_\beta,B} + \|\nabla\boldsymbol{u}_n(t)\|_{\Psi_\beta,\Omega_n}\|\boldsymbol{\varphi}\|_{\infty,B} + \|\boldsymbol{u}_n(t)\|_{\infty,\Omega_n}\|\operatorname{div}\boldsymbol{\varphi}\|_{\Psi_\beta,B} \leq\\
&\leq c\|\mathrm{D}\boldsymbol{u}_n(t)\|_{M,\Omega_n}\|\boldsymbol{\varphi}\|_{W_0^1 L_{\Psi_\beta}(B)},
\end{aligned}$$

where we have used Lemma 1.45 and the fact $\beta > 2$, thus

$$(*) \leq c\|\psi\|_{\Phi_1}\|\rho_n\boldsymbol{u}_n - \rho\boldsymbol{u}\|_{C(\langle 0,T\rangle;W^{-1}L_{\Phi_\beta}(B))}\|\mathrm{D}\boldsymbol{u}\|_{L_M(0,T;L_M(\Omega_m))}\|\boldsymbol{\varphi}\|_{W_0^1 L_{\Psi_\beta}(B)},$$

which converges to zero according to (5.7). For the convergence of the second integral we use weak-$*$ convergence (5.3). It is possible because

$$\left|\int_0^T \left(\int_B \rho_n\boldsymbol{u}_n\cdot\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^2\mathrm{d}t\right| \leq \int_0^T \|\rho_n(t)\|_{\Phi_\beta}^2\|\boldsymbol{u}_n(t)\|_\infty^2\|\boldsymbol{\varphi}\|_{\Psi_\beta}^2\,\mathrm{d}t \leq$$

$$\leq \|\rho_n(t)\|^2_{L^\infty(0,T;L_{\Phi_\beta}(B))}\|\boldsymbol{u}_n(t)\|^2_{L^2(0,T;L^\infty(B))}\|\boldsymbol{\varphi}\|^2_{\Psi_\beta},$$

and thus $\rho_n\boldsymbol{u}_n$ arew bounded in $L^2(0,T;L_{\Phi_\beta}(B))$. Altogether

$$\rho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n \overset{*}{\rightharpoonup} \rho\boldsymbol{u}\otimes\boldsymbol{u} \quad \text{in } L_M(0,T;W^{-1}L_{\Phi_\beta}(B)),$$

i.e. from uniqueness of the limit function

$$\rho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n \overset{*}{\rightharpoonup} \rho\boldsymbol{u}\otimes\boldsymbol{u} \quad \text{in } L^q(0,T;L_{\Phi_\beta}(B))$$

for $q \in [1,\infty)$.

In the case of $\rho_n \boldsymbol{f}_n$ we have

$$\left| \int_0^T \psi(t) \int_B (\rho_n \boldsymbol{f}_n - \rho \boldsymbol{f}) \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right| = \left| \int_0^T \psi(t) \int_B (\rho_n - \rho) \boldsymbol{f} \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right|$$

and we can apply weak-$*$ convergence (5.1).

Next obviously $\mathbf{S}(\mathrm{D}\boldsymbol{u}_n) \xrightarrow{M} \overline{\mathbf{S}(\mathrm{D}\boldsymbol{u})}$ in $L_{\overline{M}}(B \times (0,T))$ from (4.3) and the properties of $\mathbf{S}$.

From definition of the open set it follows that ρ, $\boldsymbol{u}$ satisfy the equation

$$\partial_t(\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla \rho - \operatorname{div} \overline{\mathbf{S}(\mathrm{D}\boldsymbol{u})} = \rho \boldsymbol{f} \quad \text{in } \mathcal{D}'(\Omega \times (0,T)). \tag{5.8}$$

Similarly to [9, str. 62] we prove that couple $(\rho, \boldsymbol{u})$ satisfies in the sense of distributions the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \rho |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x} + \int_\Omega \overline{\mathbf{S}(\mathrm{D}\boldsymbol{u})} : \mathrm{D}\boldsymbol{u} \, \mathrm{d}\boldsymbol{x} - \int_\Omega \rho \operatorname{div} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \rho \boldsymbol{u} \cdot \boldsymbol{f} \, \mathrm{d}\boldsymbol{x}. \tag{5.9}$$

Nevertheless, for ρ_n, $\boldsymbol{u}_n$ we have (see Theorem 2.2)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_n} \frac{1}{2} \rho_n |\boldsymbol{u}_n|^2 \, \mathrm{d}\boldsymbol{x} + \int_{\Omega_n} \mathbf{S}(\mathrm{D}\boldsymbol{u}_n) : \mathrm{D}\boldsymbol{u}_n \, \mathrm{d}\boldsymbol{x} - \int_{\Omega_n} \rho_n \operatorname{div} \boldsymbol{u}_n \, \mathrm{d}\boldsymbol{x} = \int_{\Omega_n} \rho_n \boldsymbol{u}_n \cdot \boldsymbol{f}_n \, \mathrm{d}\boldsymbol{x}. \tag{5.10}$$

After subtraction of the identities we obtain

$$\begin{aligned}
&\int_0^\tau \varphi_h(t) \left(\int_{\Omega_n} \mathbf{S}(\mathrm{D}\boldsymbol{u}_n) : \mathrm{D}\boldsymbol{u}_n \, \mathrm{d}\boldsymbol{x} - \int_\Omega \mathbf{S}(\mathrm{D}\boldsymbol{u}) : \mathrm{D}\boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \right) \mathrm{d}t = \\
&= \int_0^\tau \varphi_h'(t) \left(\int_{\Omega_n} \frac{1}{2} \rho_n |\boldsymbol{u}_n|^2 \, \mathrm{d}\boldsymbol{x} - \int_\Omega \frac{1}{2} \rho |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x} \right) \mathrm{d}t + \\
&\quad + \int_0^\tau \varphi_h(t) \left(\int_{\Omega_n} \rho_n \operatorname{div} \boldsymbol{u}_n \, \mathrm{d}\boldsymbol{x} - \int_\Omega \rho \operatorname{div} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \right) \mathrm{d}t + \\
&\quad + \int_0^\tau \varphi_h(t) \left(\int_{\Omega_n} \rho_n \boldsymbol{f}_n \cdot \boldsymbol{u}_n \, \mathrm{d}\boldsymbol{x} - \int_\Omega \rho \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \right) \mathrm{d}t,
\end{aligned}$$

where $\varphi_h \in C_0^\infty(0,T)$, $0 \le \varphi_h \le 1$, $\varphi_h \uparrow 1$ for $h \to 0$ a.e. in $[0,T]$. Let Q, $\overline{Q} \subset \Omega$, be a suitable set generated by a finite unification of balls $B \subset \overline{B} \subset \Omega$. We treat the above integrals one by one

$$\int_0^\tau \varphi_h'(t)\left(\int_{\Omega_n}\frac{1}{2}\rho_n|\boldsymbol{u}_n|^2\,\mathrm{d}\boldsymbol{x}-\int_\Omega\frac{1}{2}\rho|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x}\right)\,\mathrm{d}t=$$
$$=\int_0^\tau \varphi_h'(t)\left(\int_{\Omega_n\setminus Q}\frac{1}{2}\rho_n|\boldsymbol{u}_n|^2\,\mathrm{d}\boldsymbol{x}-\int_{\Omega\setminus Q}\frac{1}{2}\rho|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x}\right)\,\mathrm{d}t+$$
$$+\int_0^\tau \varphi_h'(t)\int_Q\frac{1}{2}\left(\rho_n|\boldsymbol{u}_n|^2-\rho|\boldsymbol{u}|^2\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

where

$$\int_0^\tau \varphi_h'(t)\int_Q\frac{1}{2}\left(\rho_n|\boldsymbol{u}_n|^2-\rho|\boldsymbol{u}|^2\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\to 0 \text{ for } n\to\infty$$

from the weak-$*$ convergence of $\rho_n|\boldsymbol{u}_n|^2$ in $L^q(0,T;L_{\Phi_\beta}(Q))$, see (5.6), and

$$\int_0^\tau \varphi_h'(t)\left(\int_{\Omega_n\setminus Q}\frac{1}{2}\rho_n|\boldsymbol{u}_n|^2\,\mathrm{d}\boldsymbol{x}-\int_{\Omega\setminus Q}\frac{1}{2}\rho|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x}\right)\,\mathrm{d}t\le$$
$$\le c(h)\Big(\|\chi_{\Omega_n\setminus Q}\|_{\Psi_\beta}+\|\chi_{\Omega\setminus Q}\|_{\Psi_\beta}\Big)<\varepsilon \qquad ,$$

as a consequence of (see [7, page 136]), the boundedness $\rho_n|\boldsymbol{u}_n|^2$ in $L^q(0,T;L_{\Phi_\beta}(Q))$ and the suitable choice of Q depending on h. We argue similarly for

$$\int_0^\tau \varphi_h(t)\left(\int_{\Omega_n}\rho_n\boldsymbol{f}_n\cdot\boldsymbol{u}_n\,\mathrm{d}\boldsymbol{x}-\int_\Omega\rho\boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\right)\,\mathrm{d}t.$$

For the remaining term we use the same method as at the beginning of Section 3 to derive from the continuity equations the identities

$$\int_{\Omega_n}\Big(\rho_n(\tau)\ln(\rho_n(t)+\delta)-\rho_0^n\ln(\rho_0^n+\delta)\Big)\,\mathrm{d}\boldsymbol{x}=-\int_0^\tau\int_{\Omega_n}\frac{\rho_n^2}{\delta+\rho_n}\operatorname{div}\boldsymbol{u}_n\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

and

$$\int_\Omega\Big(\rho(\tau)\ln(\rho(t)+\delta)-\rho_0\ln(\rho_0+\delta)\Big)\,\mathrm{d}\boldsymbol{x}=-\int_0^\tau\int_\Omega\frac{\rho^2}{\delta+\rho}\operatorname{div}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

where $\delta \in (0,1)$. Now we can take a ball B, $\Omega \cup \Omega_n \subset \overline{B}$. From zero prolongation of ρ and ρ_n and the convexity of functional $\int_B \rho \ln(\rho+\delta)\,\mathrm{d}\boldsymbol{x}$ we have

$$\begin{aligned}
&\int_\Omega \rho(\tau)\ln(\rho(t)+\delta)\,\mathrm{d}\boldsymbol{x} - \int_{\Omega_n} \rho_n(\tau)\ln(\rho_n(t)+\delta)\,\mathrm{d}\boldsymbol{x} = \\
&= \int_B \Big(\rho(\tau)\ln(\rho(t)+\delta) - \rho_n(\tau)\ln(\rho_n(t)+\delta)\Big)\,\mathrm{d}\boldsymbol{x} \le \\
&\le \int_B \left(\ln(\rho+\delta) + \frac{\rho}{\rho+\delta}\right)(\rho-\rho_n)\,\mathrm{d}\boldsymbol{x} \to 0 \text{ for } n\to\infty
\end{aligned}$$

as a consequence of (5.4), and

$$\begin{aligned}
&\int_{\Omega_n} \rho_0^n \ln(\rho_0^n+\delta)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \rho_0\ln(\rho_0+\delta)\,\mathrm{d}\boldsymbol{x} = \\
&= \int_{\Omega_n} \rho_0\ln(\rho_0+\delta)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \rho_0\ln(\rho_0+\delta)\,\mathrm{d}\boldsymbol{x} \le c(n),
\end{aligned}$$

where $c(n)\to 0$, $n\to\infty$. It follows from the foregoing estimates

$$\begin{aligned}
&\int_0^\tau \varphi_h(t)\left(\int_{\Omega_n}\rho_n \operatorname{div}\boldsymbol{u}_n\,\mathrm{d}\boldsymbol{x} - \int_\Omega \rho\operatorname{div}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\right)\mathrm{d}t = \\
&= \int_0^\tau (\varphi_h(t)-1)\int_{\Omega_n}\rho_n\operatorname{div}\boldsymbol{u}_n\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau(\varphi_h(t)-1)\int_\Omega\rho\operatorname{div}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \\
&\quad + \int_0^\tau\!\!\int_{\Omega_n}\frac{\delta\rho_n}{\delta+\rho_n}\operatorname{div}\boldsymbol{u}_n\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\!\!\int_\Omega\frac{\delta\rho}{\delta+\rho}\operatorname{div}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \\
&\quad + \int_0^\tau\!\!\int_{\Omega_n}\frac{\rho_n^2}{\delta+\rho_n}\operatorname{div}\boldsymbol{u}_n\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\!\!\int_\Omega\frac{\rho^2}{\delta+\rho}\operatorname{div}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le \\
&\le \|\varphi_h-1\|_{\overline{M}}\|\rho_n\|_{L^\infty(0,T;L_{\Phi_\beta}(\Omega_n))}\|\mathrm{D}\boldsymbol{u}_n\|_{L_M(0,T;L_M(\Omega_n))} + \\
&\quad + \|\varphi_h-1\|_{\overline{M}}\|\rho\|_{L^\infty(0,T;L_{\Phi_\beta}(\Omega))}\|\mathrm{D}\boldsymbol{u}\|_{L_M(0,T;L_M(\Omega))} + \\
&\quad + \delta\|\mathrm{D}\boldsymbol{u}_n\|_{L_M(0,T;L_M(\Omega_n))} + \delta\|\mathrm{D}\boldsymbol{u}\|_{L_M(0,T;L_M(\Omega))} + c(n) \le \\
&\le K(c_1(h)+c_2(n)+\delta)
\end{aligned}$$

where $c_1(h)\to 0$ for $h\to 0$, $c_2(n)\to 0$ for $n\to\infty$ and δ can be arbitrarily small.

Altogether

$$\liminf_{n\to\infty}\int_0^\tau \varphi_h(t)\int_{\Omega_n} \boldsymbol{S}(\mathrm{D}\boldsymbol{u}_n):\mathrm{D}\boldsymbol{u}_n\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t-\int_0^\tau \varphi_h(t)\int_\Omega \boldsymbol{S}(\mathrm{D}\boldsymbol{u}):\mathrm{D}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\le$$
$$\le c_1(h)+c_2(h) \tag{5.11}$$

Now we take a sequence of open sets Q_k defined similarly as Q such that $\overline{Q_k}\subset\Omega$, $|\Omega\setminus Q_k|\to 0$, and functions

$$\psi_k\in C_0^\infty(Q_k),\quad 0\le\psi_k\le 1,\quad \psi_k\uparrow 1 \text{ a.e. in } \Omega.$$

From the monotonicity of the stress tensor $\boldsymbol{S}$ we infer

$$\int_0^\tau \varphi_h(t)\int_{\Omega_n} (\boldsymbol{S}(\mathrm{D}\boldsymbol{u}_n)-\boldsymbol{S}(\mathrm{D}\boldsymbol{v})):(\mathrm{D}\boldsymbol{u}_n-\mathrm{D}\boldsymbol{v})\psi_k\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\ge 0,$$

i.e.

$$\int_0^\tau \varphi_h(t)\int_{\Omega_n} \boldsymbol{S}(\mathrm{D}\boldsymbol{u}_n):\mathrm{D}\boldsymbol{u}_n\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\ge\int_0^\tau \varphi_h(t)\int_{\Omega_n} \boldsymbol{S}(\mathrm{D}\boldsymbol{u}_n):\mathrm{D}\boldsymbol{u}_n\psi_k\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\ge$$
$$\ge\int_0^\tau \varphi_h(t)\int_{\Omega_n} \Big(\boldsymbol{S}(\mathrm{D}\boldsymbol{u}_n):\mathrm{D}\boldsymbol{v}+\boldsymbol{S}(\mathrm{D}\boldsymbol{v}):\mathrm{D}\boldsymbol{u}_n-\boldsymbol{S}(\mathrm{D}\boldsymbol{v}):\mathrm{D}\boldsymbol{v}\Big)\psi_k\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

and thus letting $n\to\infty$ and using (5.11)

$$\int_0^\tau \varphi_h(t)\int_\Omega \overline{\boldsymbol{S}(\mathrm{D}\boldsymbol{u})}:\mathrm{D}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\ge$$
$$\ge\int_0^\tau \varphi_h(t)\int_\Omega \Big(\overline{\boldsymbol{S}(\mathrm{D}\boldsymbol{u})}:\mathrm{D}\boldsymbol{v}+\boldsymbol{S}(\mathrm{D}\boldsymbol{v}):\mathrm{D}\boldsymbol{u}-\boldsymbol{S}(\mathrm{D}\boldsymbol{v}):\mathrm{D}\boldsymbol{v}\Big)\psi_k\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

In view of the Lebesgue theorem for $k\to\infty$

$$\int_0^\tau \varphi_h(t)\int_\Omega (\overline{\boldsymbol{S}(\mathrm{D}\boldsymbol{u})}-S(\mathrm{D}\boldsymbol{v})):(\mathrm{D}\boldsymbol{u}-\mathrm{D}\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\ge -c(\delta,h).$$

Since $c(\delta,h)$ is arbitrarily small we get using the Lebesgue theorem

$$\int_0^\tau\int_\Omega (\overline{\boldsymbol{S}(\mathrm{D}\boldsymbol{u})}-S(\mathrm{D}\boldsymbol{v})):(\mathrm{D}\boldsymbol{u}-\mathrm{D}\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\ge 0.$$

Then put $\boldsymbol{v} = \boldsymbol{u} - \lambda\psi$, where $\psi \in C_0^\infty(\Omega \times (0,T))$ and $\lambda > 0$. As a consequence to the monotonicity and the third property of the stress tensor $\mathcal{S}$ we infer

$$\operatorname{div} \overline{S(\mathrm{D}\boldsymbol{u})} = \operatorname{div} \mathcal{S}(\mathrm{D}\boldsymbol{u}) \quad \text{in } L_{\Phi_{\frac{1}{\alpha}}}(0,T; W^{-1}L_{\Phi_{\frac{1}{2}}}(\Omega)), \quad \alpha > 2.$$

Moreover, $\mathcal{S}(\mathrm{D}\boldsymbol{u}) \in L_{\overline{M}}(\Omega \times (0,T))$ in view of (2.6), (2.8) and (2.9).

5.3 Energy inequality

In the same way as in [9] we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{1}{2}\rho|\boldsymbol{u}|^2 + \rho \ln \rho \,\mathrm{d}\boldsymbol{x}\right) + \int_\Omega \mathcal{S}(\mathrm{D}\boldsymbol{u}) : \mathrm{D}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho\boldsymbol{f} \cdot \boldsymbol{u}\,\mathrm{d}\boldsymbol{x}$$

The energy inequality (2.11) can be derived similarly as in [1].

Conclusion

In this Thesis we studied the behaviour of the variational solutions to Navier-Stokes equations describing viscous compressible isothermal fluids with nonlinear stress tensors in a sequence of domains $\{\Omega_n\}_{n=1}^{\infty}$ which converges to a domain Ω. We proved that the solutions converge to a solution of the corresponding Navier-Stokes equations in Ω. The proof is based on the application of the theory of Orlicz spaces. We also had to prove some basic lemmas for the Orlicz spaces of Bochner's type. We genaralized the existence-result from [4], [5] and [9], where $C^{2+\mu}$-regularity of the boundary of the domain was required. The developed technique can be applied to the shape optimization of the respective fluids.

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